ISR 2026 — Basic Track

Jörg Endrullis

Cynthia Kop

Femke van Raamsdonk

partially based on slides by: Aart Middeldorp

hosting institution:
Radboud Universiteit Nijmegen
The Netherlands

- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

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and we sometimes add a binary symbol mul

more terms: $\mathsf{mul}(0, x)$ $\mathsf{mul}(\mathsf{s}(0), \mathsf{add}(0, \mathsf{s}(0)))$

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Remark

It is helpful to draw terms as trees.

Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

some terms $\quad x \cdot x^{-} \quad x^{-} \cdot (x \cdot y) \quad \mathsf{e} \cdot \mathsf{e}$

we sometimes keep the infix notation

and sometimes even use postfix

Example (Addition on Natural Numbers in Decimal Notation)

signature	$0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
some terms	$1 + 3 \quad 2 + (7 : 3) \quad (2 : (3 : x)) + ((1 + 7) : 2)$
	$(2 : x) + ((1 : x) : y)$
	also here we use infix notation

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

some terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

this is an example of an applicative TRS

we have one binary function symbol for application,

and further only constants

also, it is a very famous TRS

inventor Moses Schönfinkel (1924)

Example (Chameleons)

we can see the expressions as words, or strings

alphabet $\mathsf{R}, \mathsf{B}, \mathsf{G}$

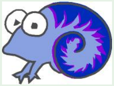
some words R $\mathsf{R\,R\,G}$ $\mathsf{R\,G\,B\,B}$

words can be seen as first-order terms

signature $\mathsf{R}, \mathsf{B}, \mathsf{G}$ all unary

terms $\mathsf{R}(x)$ $\mathsf{R}(\mathsf{R}(\mathsf{G}(x)))$ $\mathsf{R}(\mathsf{G}(\mathsf{B}(\mathsf{B}(x))))$

often the word notation is more convenient



Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

this is not a first-order TRS

because of the presence of bound variables $\lambda x.\, x\, y$

inventor Alonzo Church (1936)

string rewriting intuitively

The rewrite rules

$$\begin{array}{rcl} \mathsf{R\,G} & \to & \mathsf{B\,B} \\ \mathsf{G\,R} & \to & \mathsf{B\,B} \end{array}$$

induce the rewrite relation $\to$ on the set of all words.
We use can a rule *anywhere in a string*, for example:

$$\mathsf{R\,B\,{\color{red}R}\,{\color{green}G}\,R\,G} \to \mathsf{R\,B\,{\color{blue}B}\,{\color{blue}B}\,R\,G}$$

term rewriting intuitively

The rewrite rules

$$\begin{array}{rcl} \mathsf{add}(0, y) & \rightarrow & y \\ \mathsf{add}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{add}(x, y)) \end{array}$$

act as schemes: we get a step by *instantiating the variables*:

$$\mathsf{add}(\mathsf{s}(0), \mathsf{s}(\mathsf{add}(z, 0))) \rightarrow \mathsf{s}(\mathsf{add}(0, \mathsf{s}(\mathsf{add}(z, 0))))$$

Also, we can do such a step *anywhere in a term*:

$$\mathsf{add}(\mathsf{add}(\mathsf{s}(0), \mathsf{s}(\mathsf{add}(z, 0))), \mathsf{s}(0)) \rightarrow \mathsf{add}(\mathsf{s}(\mathsf{add}(0, \mathsf{s}(\mathsf{add}(z, 0)))), \mathsf{s}(0))$$

so we get ARSs

- The objects are strings over the alphabet $\{\mathsf{R}, \mathsf{B}, \mathsf{G}\}$, and the rewrite relation $\rightarrow$ on the objects is induced by the rewrite rules.
- The objects are first-order terms over the signature $0, \mathsf{s}, \mathsf{add}$, and the rewrite relation $\rightarrow$ on those terms is induced by the rewrite rules.

we will make more precise

- instantiate the variables by the notion of substitution

$$x \mapsto \mathsf{s}(0) \quad y \mapsto \mathsf{add}(z, 0)$$

- anywhere in a string or term by the notion of context or position

$$\mathsf{R}\,\mathsf{B}\,\square\,\mathsf{R}\,\mathsf{G}$$

$$\mathsf{add}(\square, \mathsf{s}(0))$$

so: back to the statics, the terms

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    - $s = t$

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term $(2 : x) + ((1 : x) : y)$ has subterms

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Definition (Positions)

- $\mathcal{P}\mathrm{os}(\,\cdot\,)$, the set of positions of a term

$$\mathcal{P}\mathrm{os}(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{X} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}\mathrm{os}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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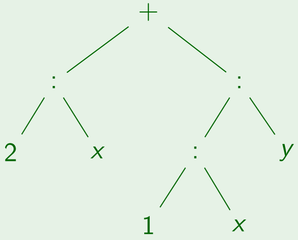

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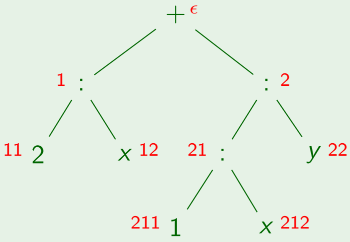

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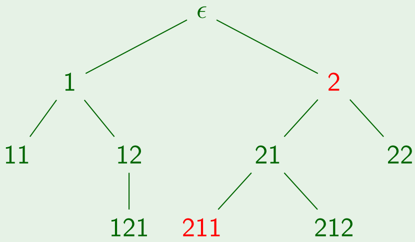

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- $p < q$ if $\exists\, r \neq \epsilon\colon pr = q$ "p is strictly above q" "q is strictly below p"
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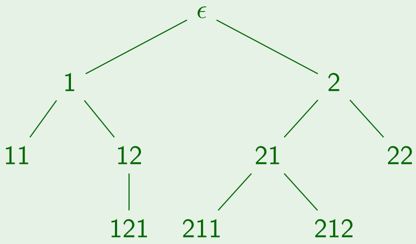

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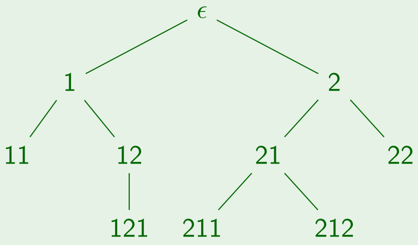

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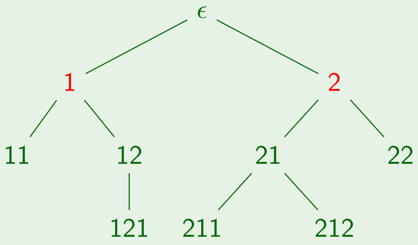

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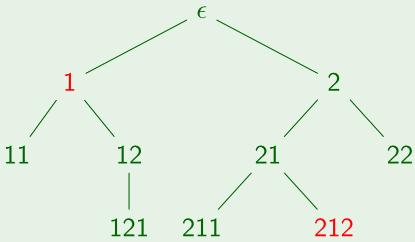

- $2 < 211$
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Definitions

- $t|_p$ subterm of t at position p

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_q & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

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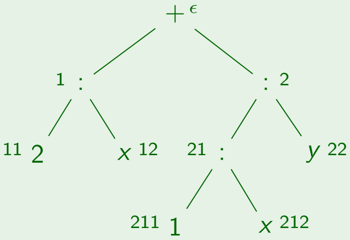
- $t[s]_p$ replace subterm in t at position p by s

$$t[s]_p = \begin{cases} s & \text{if } p = \epsilon \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

Example

$t = (2 : x) + ((1 : x) : y)$



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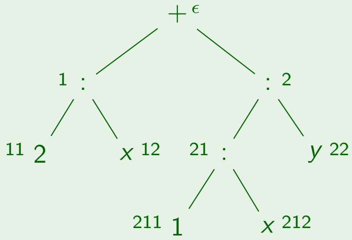

- $t|_{21} = ?$

Example

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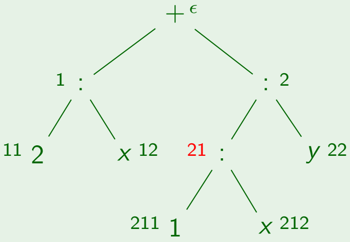

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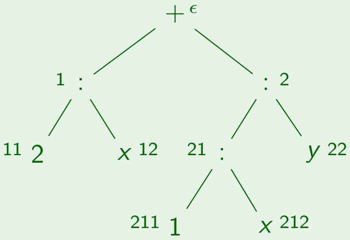

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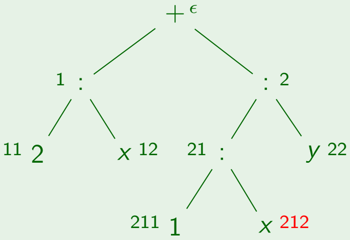

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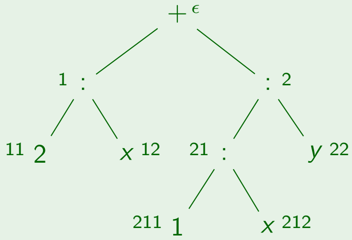

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- $t[x + 3]_2 = ?$

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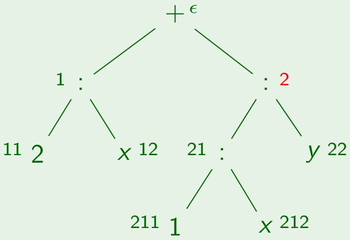

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- $t[x + 3]_2 = (2 : x) + (x + 3)$

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- $\Box[\mathsf{s}(0)] = \mathsf{s}(0) \qquad (\Box + x)[0 + x] = (0 + x) + x$

Definitions

- context is term with one hole $\Box$,
  i.e., element of $\mathcal{T}(\Sigma \cup \{\Box\}, \mathcal{X})$ that contains exactly one occurrence of $\Box$
- $C[t]$ denotes result of replacing hole in context C by term t
- relation R on terms is closed under contexts if $\forall$ terms s, t

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- substitution is mapping $\sigma\colon \mathcal{X} \to \mathcal{T}(\Sigma, \mathcal{X})$ such that its domain

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$t = x + \mathsf{s}(y + z) \quad \sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\} \quad t\sigma = \mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)$

Outline

Definitions

- A rewrite rule ($\ell \to r$) is a pair (ℓ, r) of terms such that
  - $\ell \notin \mathcal{X}$ (the lhs is not a variable)
  - $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(\ell)$ (all variables in the rhs occur in the lhs)

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Example

TRS (Σ, R) with signature Σ

$$\mathsf{0}\ \text{(constant)} \qquad \mathsf{s}\ \text{(unary)} \qquad \mathsf{add}\ \text{(binary)}$$

and rewrite rules R

$$\mathsf{add}(\mathsf{0}, y) \to y$$
$$\mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

Definition (Term Rewriting)

Let $\mathcal{R} = (\Sigma, R)$ be a TRS. The rewrite relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\Sigma, \mathcal{X})$ is defined as:

$$C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma]$$

for every:

- rule $\ell \to r \in R$,
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Let p be the position of $\Box$ in C. Then we say that the rewrite step is at position p.

Example

The rule $\mathsf{add}(0, y) \rightarrow y$ with $y \mapsto \mathsf{s}(0)$ and $C = \mathsf{add}(0, \Box)$ gives rise to the step:

$$\mathsf{add}(0, \mathsf{add}(0, \mathsf{s}(0))) \rightarrow \mathsf{add}(0, \mathsf{s}(0))$$

at position 2.

Lemma

The rewrite relation $\rightarrow_{\mathcal{R}}$ is the smallest relation on $\mathcal{T}(\Sigma, \mathcal{X})$ that:

- *contains the rules R,*
- *is closed under context, and*
- *is closed under substitutions.*

That is, $\rightarrow_{\mathcal{R}}$ is the closure of R under contexts and substitutions.

Example

rules (which implicitly give the signature):

$$\begin{array}{rcl} \mathsf{add}(0, y) & \to & y \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \end{array}$$

reduce the term $\mathsf{add}(\mathsf{s}(\mathsf{s}(0)), \mathsf{s}(\mathsf{s}(0)))$

reduce the term $\mathsf{add}(0, \mathsf{s}(\mathsf{add}(\mathsf{s}(0), 0)))$

a reduction graph of a term t
has as nodes all reducts of t, and there is an arrow between s and s' if there is a reduction step from s to s'

Properties of Term Rewriting Systems

We consider TRSs $\mathcal{R} = (\Sigma, R)$ as

$$\text{ARSs } \langle \mathcal{T}(\Sigma, \mathcal{X}), \rightarrow_{\mathcal{R}} \rangle$$

TRSs inherit the properties: CR, WCR, SN, WN, NF, UN, $\mathrm{UN}^{\rightarrow}$, ...

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Outline

a lot can happen during rewriting (1/2)

Example

rules:
$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{a} \\ \mathsf{b} &\to \mathsf{a} \end{aligned}$$

step: $\mathsf{f}(\mathsf{b}) \to \mathsf{a}$

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rules:
$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(x, x) \\ \mathsf{b} &\to \mathsf{b} \end{aligned}$$

step: $\mathsf{f}(\mathsf{b}) \to \mathsf{f}(\mathsf{b}, \mathsf{b})$

a redex is duplicated

a lot can happen during rewriting (2/2)

Example

rules:

$$\begin{aligned} \mathsf{e}(x,x) &\to \mathsf{t} \\ \infty &\to \mathsf{s}(\infty) \end{aligned}$$

step $\mathsf{e}(\infty,\infty) \to \mathsf{e}(\infty,\mathsf{s}(\infty))$

the equality check does not succeed anymore

Example (Termination)

TRS $\mathcal{R}$:

$$\mathsf{f}(0, 1, x) \to \mathsf{f}(x, x, x)$$

The TRS $\mathcal{R}$ is SN, even if x is tripled. Why?

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The TRS $\mathcal{S}$ is SN. Why?

The combination $\mathcal{R} \oplus \mathcal{S}$ is not SN:

$$\begin{aligned}\mathsf{f}(\mathsf{or}(0, 1), \mathsf{or}(0, 1), \mathsf{or}(0, 1)) &\to \mathsf{f}(0, \mathsf{or}(0, 1), \mathsf{or}(0, 1)) \\ &\to \mathsf{f}(0, 1, \mathsf{or}(0, 1)) \\ &\to \mathsf{f}(\mathsf{or}(0, 1), \mathsf{or}(0, 1), \mathsf{or}(0, 1))\end{aligned}$$

By Yoshihito Toyama.

Example (Normal Form property)

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The TRS $\mathcal{S}$ has the property NF. Why?

The combination $\mathcal{R} \oplus \mathcal{S}$ does not have the property NF:

$$\mathsf{f}(\mathsf{b}, \mathsf{c}) \leftarrow \mathsf{f}(\mathsf{a}, \mathsf{c}) \leftarrow \mathsf{f}(\mathsf{a}, \mathsf{a}) \to n$$

By Aart Middeldorp.

Example (Unique Normal Forms)

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instance: a pair of terms l, t

question: $\exists$ substitution σ : $l\sigma = t$?

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Remarks

- term t can be rewritten if left-hand side of rewrite rule matches subterm of t
- matching problem is decidable (in linear time)

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$$\{x \mapsto t\} \cup S \;\Rightarrow\; \bot \text{ if } S \text{ contains } x \mapsto t' \text{ with } t \neq t'$$

Examples

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&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y \mapsto x+\mathsf{s}(0),\ z \mapsto z\}
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- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)$:

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$$
\begin{aligned}
&\{x + \mathsf{s}(y+z) \mapsto \mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ \mathsf{s}(y+z) \mapsto \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y+z \mapsto (x+\mathsf{s}(0))+z\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y \mapsto x+\mathsf{s}(0),\ z \mapsto z\}
\end{aligned}
$$

- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)$:

$$
\begin{aligned}
&\{x^- \cdot (x \cdot y) \mapsto (\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)\} \\
&\qquad \Longrightarrow \quad \{x^- \mapsto (\mathsf{e} \cdot x)^-,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{e} \cdot x,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\}
\end{aligned}
$$

Examples

- $x + \mathsf{s}(y+z)$ matches $\mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)$:

$$\begin{aligned}
&\{x + \mathsf{s}(y+z) \mapsto \mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ \mathsf{s}(y+z) \mapsto \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y+z \mapsto (x+\mathsf{s}(0))+z\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y \mapsto x+\mathsf{s}(0),\ z \mapsto z\}
\end{aligned}$$

- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)$:

$$\begin{aligned}
&\{x^- \cdot (x \cdot y) \mapsto (\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)\} \\
&\qquad \Longrightarrow \quad \{x^- \mapsto (\mathsf{e} \cdot x)^-,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{e} \cdot x,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{e} \cdot x,\ x \mapsto \mathsf{e} \cdot \mathsf{e},\ y \mapsto x\}
\end{aligned}$$

Examples

- $x + \mathsf{s}(y + z)$ matches $\mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)$:

$$
\begin{aligned}
&\{x + \mathsf{s}(y + z) \mapsto \mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ \mathsf{s}(y + z) \mapsto \mathsf{s}((x + \mathsf{s}(0)) + z)\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y + z \mapsto (x + \mathsf{s}(0)) + z\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{s}(y),\ y \mapsto x + \mathsf{s}(0),\ z \mapsto z\}
\end{aligned}
$$

- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)$:

$$
\begin{aligned}
&\{x^- \cdot (x \cdot y) \mapsto (\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)\} \\
&\qquad \Longrightarrow \quad \{x^- \mapsto (\mathsf{e} \cdot x)^-,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{e} \cdot x,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\qquad \Longrightarrow \quad \{x \mapsto \mathsf{e} \cdot x,\ x \mapsto \mathsf{e} \cdot \mathsf{e},\ y \mapsto x\} \\
&\qquad \Longrightarrow \quad \bot
\end{aligned}
$$