ISR 2026 — Basic Track

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The Netherlands

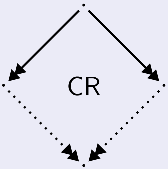
Outline

- Overview
- Orthogonality
- Beyond Orthogonality
- Developments
- Strategies

Orthogonality

Confluence

every two coinitial rewrite sequences can be joined



- . . . yields uniqueness of normal forms
- . . . is decidable for terminating TRSs

Confluence

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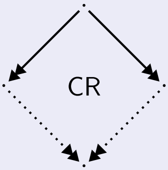

- ... yields uniqueness of normal forms
- ... is decidable for terminating TRSs
- ... what about nonterminating TRSs ?

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\mathsf{a} \rightarrow \mathsf{b}$$

$$\mathsf{a} \rightarrow \mathsf{c}$$

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$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$$
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- no critical pairs but no confluence (Klop 1978)

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{g}(x) &\to \mathsf{f}(x,\mathsf{g}(x)) \\ \mathsf{c} &\to \mathsf{g}(\mathsf{c}) \end{aligned}$$

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$$\mathsf{c} \to \mathsf{g}(\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \to \mathsf{a}$$

$$\mathsf{a} \; {}^*\!\!\leftarrow \mathsf{c} \to \mathsf{g}(\mathsf{c}) \to^* \mathsf{g}(\mathsf{a})$$

Examples (Non-Confluence)

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$$\begin{aligned} a &\to b \\ a &\to c \end{aligned} \qquad\qquad b \leftarrow a \to c$$

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$$\begin{aligned} f(x,x) &\to a \\ g(x) &\to f(x,g(x)) \\ c &\to g(c) \end{aligned} \qquad\qquad \begin{aligned} &c \to g(c) \to f(c,g(c)) \to f(g(c),g(c)) \to a \\ &a \mathrel{{}^*{\leftarrow}} c \to g(c) \to^* g(a) \end{aligned}$$

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Confluence via Critical Pairs

Control interference of rewrite rules:

- Critical Pair Lemma:
  WCR $\iff$ $\leftarrow\rtimes\rightarrow \subseteq \downarrow$ (all critical pairs are convergent)
- Combine with Newman's Lemma:
  SN & $\leftarrow\rtimes\rightarrow \subseteq \downarrow$ $\implies$ CR

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Observe from preceding examples:

Confluence via Orthogonality

Forbid interference of rewrite rules

- no critical pairs
- no equality checks (duplicated variables in the left-hand sides)

Definitions

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- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear

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Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear
- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

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A TRS is orthogonal if it is left-linear and has no criticial pairs.

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Examples

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

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((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z)
\end{aligned}
\qquad
\begin{aligned}
\mathsf{ack}(0, y) &\to \mathsf{s}(y) \\
\mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\
\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))
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Theorem

Every orthogonal TRS is confluent

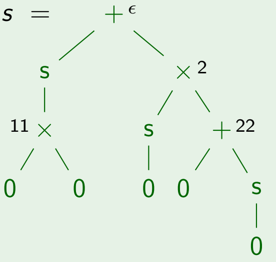
$$\begin{tikzcd} \forall s, t, u & & s \arrow[dl] \arrow[dr] & \\ & t \arrow[dr, dotted] & & u \arrow[dl, dotted] \\ \exists v & & v & \end{tikzcd}$$

For orthogonal TRSs there is a canonical way to compute common reduct. . .

Example

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

$s =$



redex positions in s: $\epsilon \mid 11 \mid 2 \mid \quad 22 \quad \mid$

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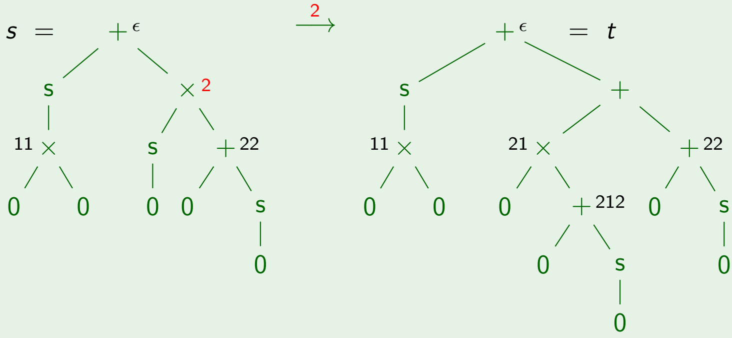

redex positions in s: ϵ | 11 | 2 | 22 |

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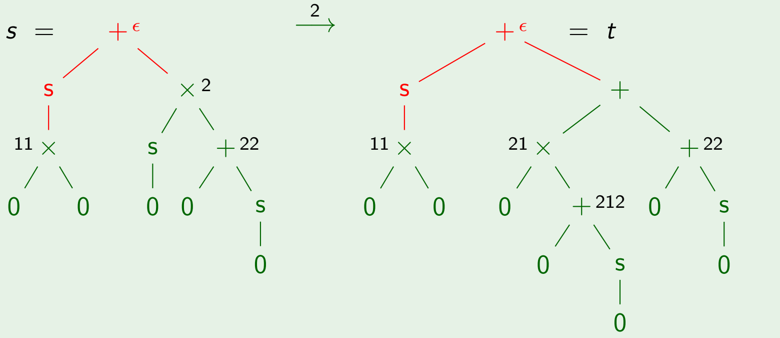
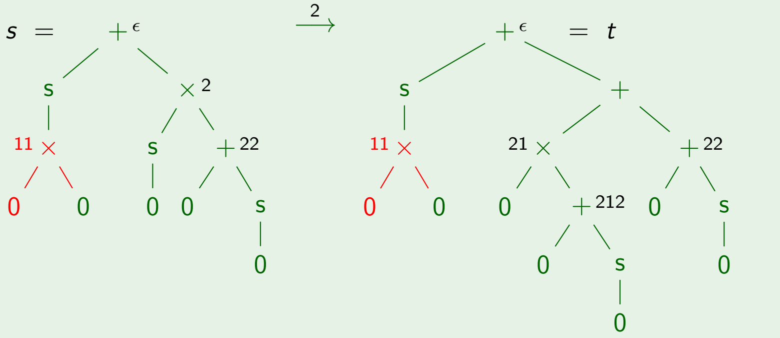


redex positions in s :	ϵ	11	2	22	
redex positions in t :	ϵ				

Example

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$$



redex positions in s:	ϵ	11	2	22	
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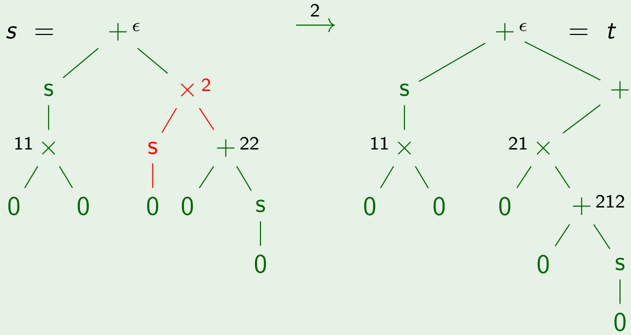

redex positions in s:	ϵ	11	2	22
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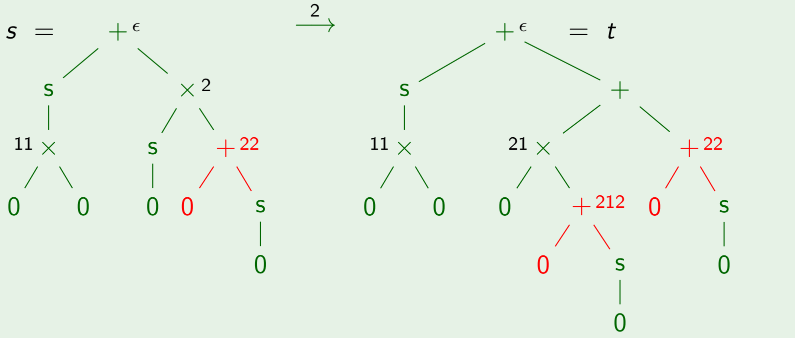

redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ	11		212	22

duplicated

Example

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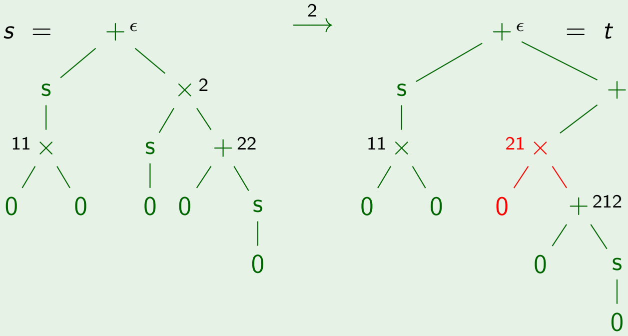

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

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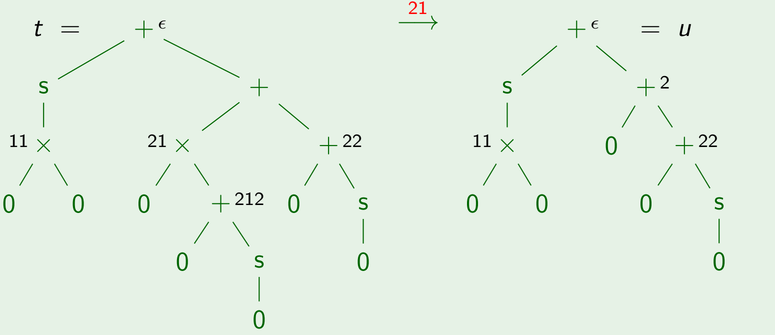

redex positions in s:	ϵ	11	2	22		
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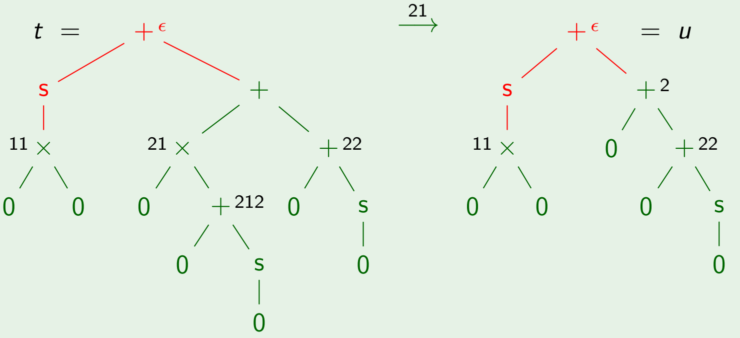

redex positions in s:	ϵ	11	2	22		
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redex positions in u:	ϵ					

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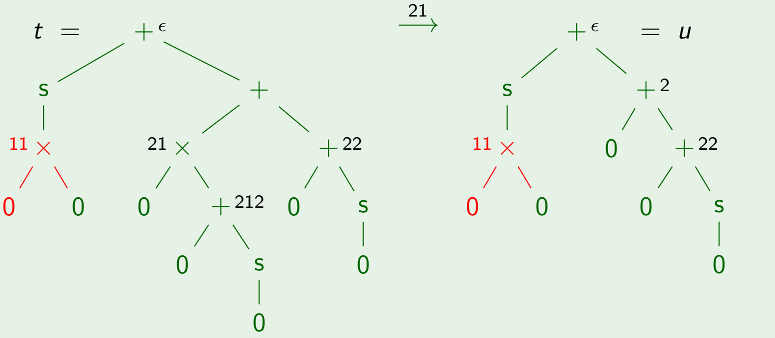

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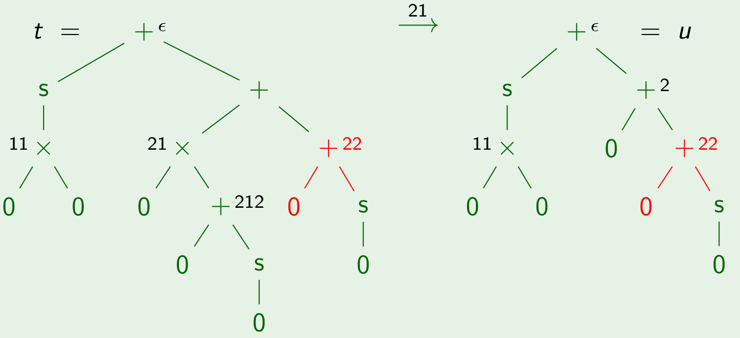

redex positions in s:	ϵ	11	2	22		
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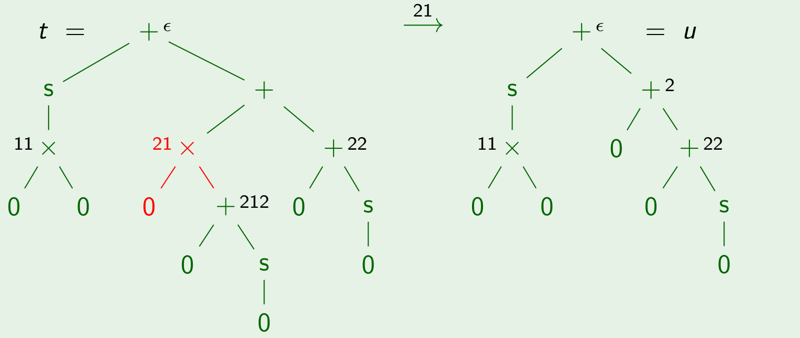

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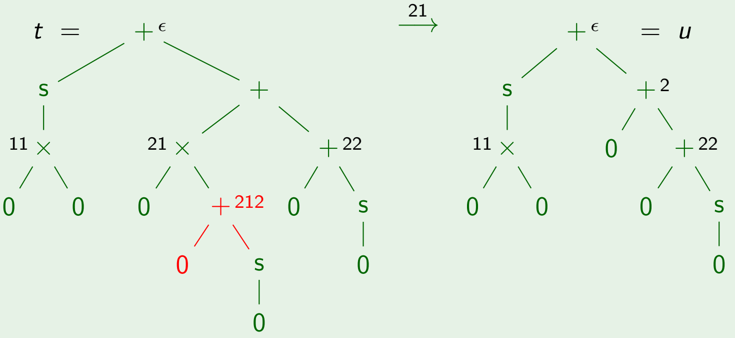

redex positions in s:	ϵ	11	2	22		
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erased

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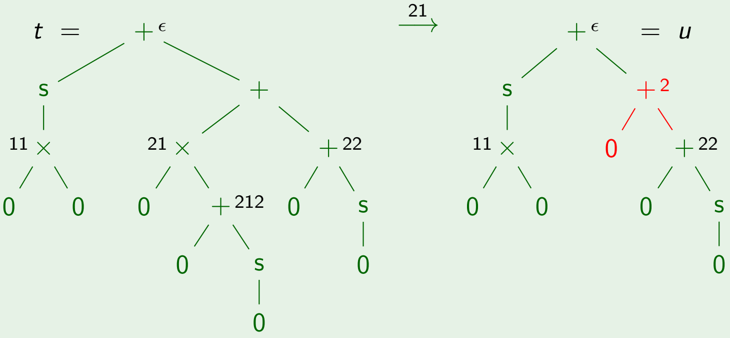

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redex at position 212 is erased redex at position 2 is created rewrite step $A \colon s \xrightarrow[\ell \to r]{p} t$ at position $p \in \mathcal{P}os(s)$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Step)

- The descendants of q after A in t

$$q/A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}os_{\mathcal{X}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

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- The descendants of Q after A in t are

$$Q/A = \bigcup_{q \in Q} q/A$$

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Sequence)

The descendants of Q after A in t

$$Q/A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q/A_1)/A_2 & \text{If } A = A_1; A_2 \text{ with } A_1\colon s \to u \text{ and } A_2\colon u \to^* t \end{cases}$$

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Lemma

For left-linear TRSs: if Q is parallel ($\forall\, p \neq q \in Q\colon p \parallel q$) then so is Q/A.

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For orthogonal TRSs: if Q is set of redex positions then so is Q/A.

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Lemma

For orthogonal TRSs: if Q is set of redex positions then so is Q/A.

A descendant of redex is called residual.

Remark

- In non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(x,x) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{f}(\mathsf{b},\mathsf{a})$$

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- In TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(\mathsf{a}) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b})$$

Definition (Parallel Rewriting)

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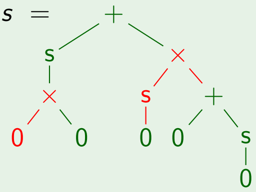
Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$

$s =$



Definition (Parallel Rewriting)

1. $= \; \subseteq \;\mathrel{+\!\!\!\!\!\!+\!\!\!\!\!\rightarrow}$
2. $\xrightarrow{\epsilon} \; \subseteq \;\mathrel{+\!\!\!\!\!\!+\!\!\!\!\!\rightarrow}$
3. $f(s_1, \ldots, s_n) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{+\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} t_i$ for each $1 \leqslant i \leqslant n$

Example

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

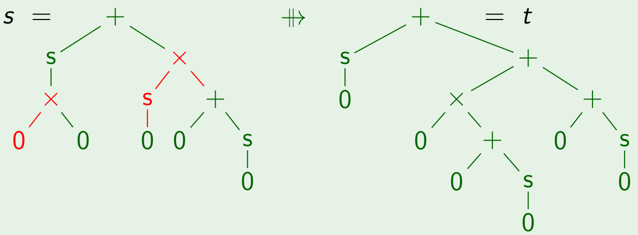
$$s = \mathsf{s}(0 \times 0) + (\mathsf{s}(0) \times (0 + \mathsf{s}(0))) \mathrel{+\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} \mathsf{s}(0) + ((0 \times (0 + \mathsf{s}(0))) + (0 + \mathsf{s}(0))) = t$$

Definition (Parallel Rewriting)

1. $= \ \subseteq \mathrel{-\!\!\Vert\!\!\rightarrow}$
2. $\xrightarrow{\epsilon} \ \subseteq \mathrel{-\!\!\Vert\!\!\rightarrow}$
3. $f(s_1, \ldots, s_n) \mathrel{-\!\!\Vert\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{-\!\!\Vert\!\!\rightarrow} t_i$ for each $1 \leqslant i \leqslant n$

Example

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$



Lemma

$s \mathrel{-\!\!\Vert\!\!\rightarrow} t \iff s \to^* t$ *by contracting redexes at pairwise parallel positions in* s

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\!\!\!\;\twoheadrightarrow} t_1$ by contracting redexes at positions in P_A

B: $s \mathrel{+\!\!\!\!\!\!\!\!\;\twoheadrightarrow} t_2$ by contracting redexes at positions in P_B

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\!\!\twoheadrightarrow} t_1$ by contracting redexes at positions in P_A

B: $s \mathrel{+\!\!\!\!\!\!\!\twoheadrightarrow} t_2$ by contracting redexes at positions in P_B

- B/A: $t_1 \mathrel{+\!\!\!\!\!\!\!\twoheadrightarrow} u_1$ by contracting redexes at positions in P_B/A

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!+\!\!\!\!\!\to} t_1$ by contracting redexes at positions in P_A

B: $s \mathrel{+\!\!\!\!\!+\!\!\!\!\!\to} t_2$ by contracting redexes at positions in P_B

- B/A: $t_1 \mathrel{+\!\!\!\!\!+\!\!\!\!\!\to} u_1$ by contracting redexes at positions in P_B/A
- A/B: $t_2 \mathrel{+\!\!\!\!\!+\!\!\!\!\!\to} u_2$ by contracting redexes at positions in P_A/B

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\!\!\;\;\to} t_1$ by contracting redexes at positions in P_A

B: $s \mathrel{+\!\!\!\!\!\!\!\;\;\to} t_2$ by contracting redexes at positions in P_B

- B/A: $t_1 \mathrel{+\!\!\!\!\!\!\!\;\;\to} u_1$ by contracting redexes at positions in P_B/A
- A/B: $t_2 \mathrel{+\!\!\!\!\!\!\!\;\;\to} u_2$ by contracting redexes at positions in P_A/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P_A

B: $s \mathrel{+\!\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in P_B

- B/A: $t_1 \mathrel{+\!\!\!\!\!\rightarrow} u_1$ by contracting redexes at positions in P_B/A
- A/B: $t_2 \mathrel{+\!\!\!\!\!\rightarrow} u_2$ by contracting redexes at positions in P_A/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Parallel Moves Lemma

For every orthogonal TRS:

$$\begin{tikzcd}
 & s \arrow[dl, "A"', "+" marking] \arrow[dr, "B", "+" marking] & \\
t_1 \arrow[d, "B/A"', "+" marking] & & t_2 \arrow[d, "A/B", "+" marking] \\
u_1 & ? & u_2
\end{tikzcd}$$

Definition (Projection)

$A\colon\ s \mathrel{+\mkern-12mu\rightarrow} t_1$ by contracting redexes at positions in P_A

$B\colon\ s \mathrel{+\mkern-12mu\rightarrow} t_2$ by contracting redexes at positions in P_B

- $B/A\colon\ t_1 \mathrel{+\mkern-12mu\rightarrow} u_1$ by contracting redexes at positions in P_B/A
- $A/B\colon\ t_2 \mathrel{+\mkern-12mu\rightarrow} u_2$ by contracting redexes at positions in P_A/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Parallel Moves Lemma

For every orthogonal TRS:

$$\begin{tikzcd}
 & s \arrow[dl, "A"'] \arrow[dr, "B"] & \\
t_1 \arrow[dr, "B/A"'] & & t_2 \arrow[dl, "A/B"] \\
 & u &
\end{tikzcd}$$

Definition (Projection)

$A\colon\ s \mathrel{\parallel\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P_A

$B\colon\ s \mathrel{\parallel\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in P_B

- $B/A\colon\ t_1 \mathrel{\parallel\!\!\!\!\rightarrow} u_1$ by contracting redexes at positions in P_B/A
- $A/B\colon\ t_2 \mathrel{\parallel\!\!\!\!\rightarrow} u_2$ by contracting redexes at positions in P_A/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Parallel Moves Lemma

For every orthogonal TRS:

$$
\begin{tikzcd}
 & s \arrow[dl, "A"', "\parallel" marking] \arrow[dr, "B", "\parallel" marking] \arrow[dd, bend right=80, "A \sqcup B"'] \arrow[dd, bend left=80, "B \sqcup A"] & \\
t_1 \arrow[dr, "B/A"', "\parallel" marking] & & t_2 \arrow[dl, "A/B", "\parallel" marking] \\
 & u &
\end{tikzcd}
$$

Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
2. $t_1 = t_2$
3. $p/A = p/B$ for all redex positions p in s_1

Definition

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Parallel Moves Lemma (with Permutation Equivalence)

$$
\begin{tikzcd}
 & s \arrow[dl, "A"'] \arrow[dr, "B"] \arrow[dd, bend right=80, "A \sqcup B"'] \arrow[dd, bend left=80, "B \sqcup A"] & \\
t_1 \arrow[dr, "B/A"'] & & t_2 \arrow[dl, "A/B"] \\
 & u &
\end{tikzcd}
$$

Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
2. $t_1 = t_2$
3. $p/A = p/B$ for all redex positions p in s_1

Parallel Moves Lemma (with Permutation Equivalence)

$$
\begin{tikzcd}
& s \arrow[dl, "A"'] \arrow[dr, "B"] \arrow[dd, bend right=90, "A \sqcup B"'] \arrow[dd, bend left=90, "B \sqcup A"] & \\
t_1 \arrow[dr, "B/A"'] & \simeq & t_2 \arrow[dl, "A/B"] \\
& u &
\end{tikzcd}
$$

Corollary

Every orthogonal TRSs is confluent.

Proof.

$$\begin{tikzcd} \cdot \arrow[rr, two heads] \arrow[d, two heads] & & \cdot \\ {} & & \end{tikzcd}$$

∎

Corollary

Every orthogonal TRSs is confluent.

Proof.

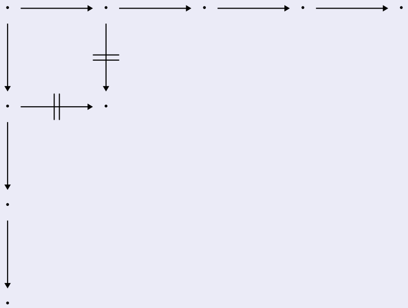
$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[d] & & & & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}
$$

∎

Corollary

Every orthogonal TRSs is confluent.

Proof.

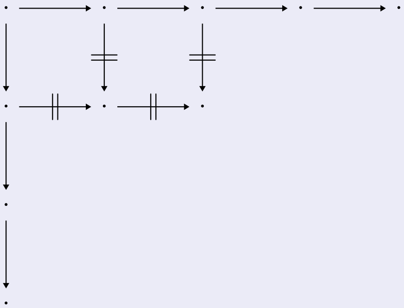


■

Corollary

Every orthogonal TRSs is confluent.

Proof.

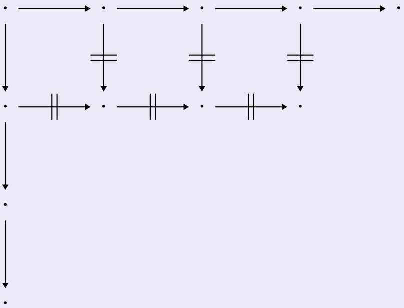


∎

Corollary

Every orthogonal TRSs is confluent.

Proof.

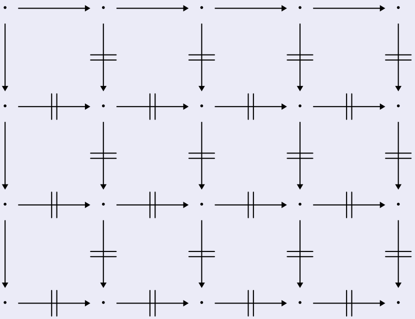


∎

Corollary

Every orthogonal TRSs is confluent.

Proof.



∎

Outline

Definition

A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

Definition

A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

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A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

Examples

$$
\begin{aligned}
x \vee \mathsf{T} &\rightarrow \mathsf{T} \\
\mathsf{T} \vee x &\rightarrow \mathsf{T} \\
\mathsf{F} \vee \mathsf{F} &\rightarrow \mathsf{F}
\end{aligned}
$$

Definition

A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

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A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

Examples

$$
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x \vee \mathsf{T} &\rightarrow \mathsf{T} \\
\mathsf{T} \vee x &\rightarrow \mathsf{T} \\
\mathsf{F} \vee \mathsf{F} &\rightarrow \mathsf{F}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{p}(\mathsf{s}(x)) &\rightarrow x \\
\mathsf{s}(\mathsf{p}(x)) &\rightarrow x
\end{aligned}
$$

Theorem

Every weakly orthogonal TRSs is confluent.

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\leftarrow\!\!\!\!\!\Vert \cdot \Vert\!\!\!\!\rightarrow \subseteq \Vert\!\!\!\!\rightarrow \cdot \leftarrow\!\!\!\!\!\Vert$

$$\begin{tikzcd}
\cdot \arrow[r, "\Vert" marking] \arrow[d, "=" marking] & \cdot \\
\cdot &
\end{tikzcd}$$

■

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\mathrel{\leftarrow\mkern-14mu\parallel} \cdot \mathrel{\parallel\mkern-14mu\rightarrow} \subseteq \mathrel{\parallel\mkern-14mu\rightarrow} \cdot \mathrel{\leftarrow\mkern-14mu\parallel}$

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[rr, "\parallel" marking] & & \cdot \\
\cdot \arrow[dd, "=" marking] & & & \\
& & & \\
\cdot & & &
\end{tikzcd}
$$

interesting case

$\blacksquare$

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\mathrel{\leftarrow\mkern-14mu\parallel} \cdot \mathrel{\parallel\mkern-14mu\rightarrow} \subseteq \mathrel{\parallel\mkern-14mu\rightarrow} \cdot \mathrel{\leftarrow\mkern-14mu\parallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & \cdot \arrow[r, "\parallel" description] & \cdot \\
\cdot \arrow[d, "=" description] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

∎

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\xleftarrow{\;\Vert\;} \cdot \xrightarrow{\;\Vert\;} \subseteq \xrightarrow{\;\Vert\;} \cdot \xleftarrow{\;\Vert\;}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & \cdot \arrow[r, "\Vert"] & \cdot \\
\cdot \arrow[urr, "\Vert"] \arrow[d, "="] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

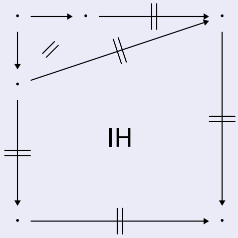
∎

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\mathrel{\twoheadleftarrow\!\!\!\!\!\!\shortparallel} \cdot \mathrel{\shortparallel\!\!\!\!\!\!\twoheadrightarrow} \subseteq \mathrel{\shortparallel\!\!\!\!\!\!\twoheadrightarrow} \cdot \mathrel{\twoheadleftarrow\!\!\!\!\!\!\shortparallel}$



interesting case

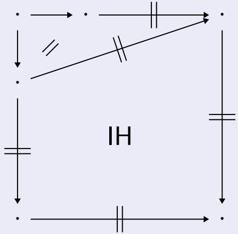
∎

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\parallel}$



interesting case

- $\rightarrow \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \subseteq \rightarrow^*$

∎

Theorem (Huet 1980)

left-linearity & $\leftarrow\rtimes\rightarrow \;\subseteq\; \nrightarrow\!\!\!\!\!\!\parallel$ $\implies$ *CR*

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\Vert\!\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\Vert\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\Vert}$

$$\begin{tikzcd} \cdot \arrow[r, "\Vert" marking] \arrow[d, "=" marking] & \cdot \\ \cdot & \end{tikzcd}$$

$\blacksquare$

Theorem (Huet 1980)

left-linearity & $\mathrel{\leftarrow\!\rtimes\!\rightarrow} \subseteq \mathrel{\Vert\!\!\!\rightarrow} \implies \mathit{CR}$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\rightarrow} \subseteq \mathrel{\Vert\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\Vert}$

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[rr, "\Vert" marking] & & \cdot \\
\cdot \arrow[dd, "=" marking] & & & \\
& & & \\
\cdot & & &
\end{tikzcd}
$$

interesting case

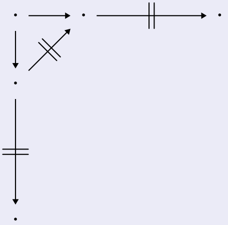
∎

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\parallel}$



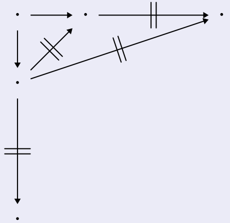
interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\parallel}$



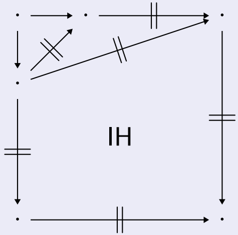
interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \quad\Longrightarrow\quad \mathit{CR}$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\parallel}$



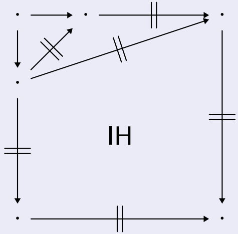
interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\rightarrow\!\!\!\!\!\!\parallel\;} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\;\parallel\!\!\!\!\!\!\leftarrow} \cdot \mathrel{\rightarrow\!\!\!\!\!\!\parallel\;} \;\subseteq\; \mathrel{\rightarrow\!\!\!\!\!\!\parallel\;} \cdot \mathrel{\;\parallel\!\!\!\!\!\!\leftarrow}$



interesting case

- $\rightarrow \;\subseteq\; \mathrel{\rightarrow\!\!\!\!\!\!\parallel\;} \;\subseteq\; \rightarrow^*$

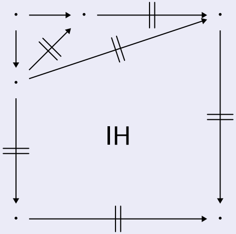
∎

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\|\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\|} \cdot \mathrel{\|\!\!\!\rightarrow} \;\subseteq\; \mathrel{\|\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\|}$



interesting case

- $\rightarrow \;\subseteq\; \mathrel{\|\!\!\!\rightarrow} \;\subseteq\; \rightarrow^*$

■

Open Problem

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\leftarrow\!\!\!\|} \quad\Longrightarrow\quad CR\ ?$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Theorem (Huet 1980)

linearity & $\leftarrow\rtimes\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

- $s \leftarrow\bowtie\rightarrow t$ for critical pair originating from overlap $\langle l_1 \to r_1, \epsilon, l_2 \to r_2 \rangle$
- $\leftarrow\!\cdot\!\rtimes\rightarrow \;=\; \leftarrow\rtimes\rightarrow \setminus \leftarrow\bowtie\rightarrow$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

- $s \leftarrow\!\bowtie\!\rightarrow t$ for critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2\rangle$
- $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\Vert\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\Vert\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

- $s \leftarrow\!\bowtie\!\rightarrow t$ for critical pair originating from overlap $\langle l_1 \to r_1, \epsilon, l_2 \to r_2 \rangle$
- $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\twoheadrightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\twoheadrightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (van Oostrom 1996)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\!\twoheadrightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\!\twoheadrightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Developments

Definition

A development of set of redex positions Q in term t is a rewrite sequence starting from t in which all contracted redex positions descend from positions in Q.

Example

- rewrite rules

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☺$$

Definition

A development of set of redex positions Q in term t is a rewrite sequence starting from t in which all contracted redex positions descend from positions in Q.

Example

- rewrite rules

$$
\begin{array}{cc}
0 + y \to y & 0 \times y \to 0 \\
\mathsf{s}(x) + y \to \mathsf{s}(x + y) & \mathsf{s}(x) \times y \to (x \times y) + y
\end{array}
$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☺$$

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to \underline{(0 \times (0 \times 0))} + (0 \times 0) \to 0 + (0 \times 0) \qquad ☹$$

Definition

A development of set of redex positions Q in term t is a rewrite sequence starting from t in which all contracted redex positions descend from positions in Q.

Example

- rewrite rules

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☺$$
$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to \underline{(0 \times (0 \times 0))} + (0 \times 0) \to 0 + (0 \times 0) \qquad ☹$$
$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0 \qquad ☺$$

Definition (Overlining)

For a TRS $\mathcal{R} = \langle \Sigma, R \rangle$ we define the overlined TRS $\overline{\mathcal{R}} = \langle \overline{\Sigma}, \overline{R} \rangle$:

- $\overline{\Sigma} = \Sigma \cup \{\overline{f} \mid f \in \Sigma\}$,
- $\overline{R} = \{\overline{\rho} \mid \rho \in R\}$

where $\overline{\rho}$ is obtained from ρ by overlining the head symbol of the left-hand side.

$$\rho : f(s_1, \ldots, s_n) \to r \quad \text{yields} \quad \overline{\rho} : \overline{f}(s_1, \ldots, s_n) \to r$$

Definition (Overlining)

For a TRS $\mathcal{R} = \langle \Sigma, R \rangle$ we define the overlined TRS $\overline{\mathcal{R}} = \langle \overline{\Sigma}, \overline{R} \rangle$:

- $\overline{\Sigma} = \Sigma \cup \{\overline{f} \mid f \in \Sigma\}$,
- $\overline{R} = \{\overline{\rho} \mid \rho \in R\}$

where $\overline{\rho}$ is obtained from ρ by overlining the head symbol of the left-hand side.

$$\rho : f(s_1, \ldots, s_n) \to r \quad \text{yields} \quad \overline{\rho} : \overline{f}(s_1, \ldots, s_n) \to r$$

Example

The overlined version of Combinatory Logic:

$$\begin{array}{lcl} \overline{Ap}(Ap(Ap(S,x),y),z) & \to & Ap(Ap(x,z),Ap(y,z)) \\ \overline{Ap}(Ap(K,x),y) & \to & x \\ \overline{Ap}(I,x) & \to & x \end{array}$$

We write $t \geq s$ if t can be obtained from s by overlining some redex positions.

Definition (Lifting)

A $\mathcal{R}$ rewrite sequence $A : s_1 \to s_2 \to \ldots \to s_n$ can be lifted if:

$$
\begin{array}{ccccccc}
t_1 & \xrightarrow{\langle \overline{\rho}_1,\, p_1 \rangle} & t_2 & \xrightarrow{\langle \overline{\rho}_2,\, p_2 \rangle} & \cdots & \xrightarrow{\langle \overline{\rho}_{n-1},\, p_{n-1} \rangle} & t_n \\
\vee\!\mid & & \vee\!\mid & & & & \vee\!\mid \\
s_1 & \xrightarrow{\langle \rho_1,\, p_1 \rangle} & s_2 & \xrightarrow{\langle \rho_2,\, p_2 \rangle} & \cdots & \xrightarrow{\langle \rho_{n-1},\, p_{n-1} \rangle} & s_n
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Lemma

For orthogonal TRSs: a reduction is a development $\iff$ *it can be lifted.*

Theorem

Properties of $\overline{\mathcal{R}}$ for orthogonal TRSs $\mathcal{R}$:

- *$\overline{\mathcal{R}}$ is orthogonal.*
- *$\overline{\mathcal{R}}$ is SN.*
- *$\overline{\mathcal{R}}$ is CR.*

Proof.

The orthogonality of $\overline{\mathcal{R}}$ is immediate. Hence $\overline{\mathcal{R}}$ is CR.

For SN we show that $\overline{\mathcal{R}}$ is ILPO terminating where $\overline{f} > g$ for every $f, g \in \Sigma$.
Let $\ell = \overline{f}(\ell_1, \ldots, \ell_n)$ and $r \in \mathcal{T}(\Sigma, \mathcal{X})$ with $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}\ell$.
Then $\ell^* \to_{ilpo}^* r$ by induction on r:

- If $r \in \mathcal{V}\mathrm{ar}(\ell)$, then we use $\to_{put}$ and $\to_{select}$.
- If $r = g(r_1, \ldots, r_m)$, then we use $\ell^* \to_{copy} g(\ell^*, \ldots, \ell^*)$.
  Moreover by the induction hypothesis $\ell^* \to_{ilpo}^* r_i$ for every i.

∎

Theorem

Developments are finite.

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Definition

A development $A\colon s \to^* t$ of $Q \subseteq \mathcal{P}\mathrm{os}(s)$ is complete if $Q/A = \varnothing$.

We write $s \mathrel{\multimap\!\!\!\!\to} t$ (called multi-step) if there is a complete development $s \to^* t$.

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Example

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☹$$

$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0 \qquad ☺$$

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Theorem

All complete developments of Q are permutation equivalent.

Strategies

Definition

A rewrite strategy $\mathcal{S}$ is mapping that assigns to every reducible term t a nonempty set of finite nonempty rewrite sequences starting from t.

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Definition

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Lemma

For terminating TRSs every strategy is normalizing and perpetual.

Example

- rewrite rules

$$
\begin{array}{llcl}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{cc}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

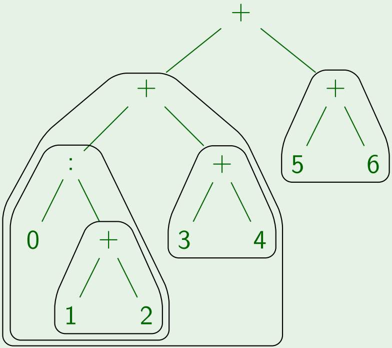
- term

$$((0:(1+2))+(3+4))+(5+6)$$

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

tree representation



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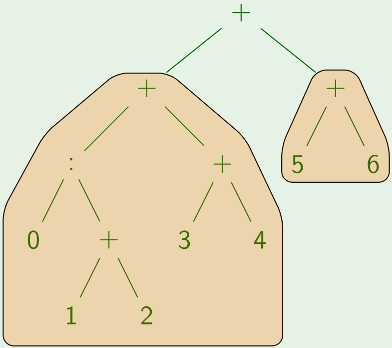

outermost redexes

Example (cont'd)

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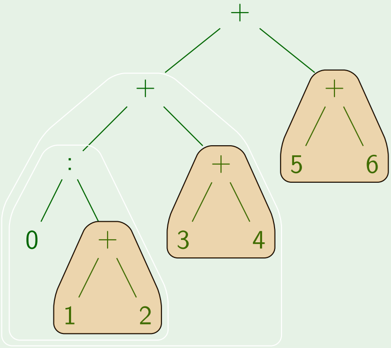
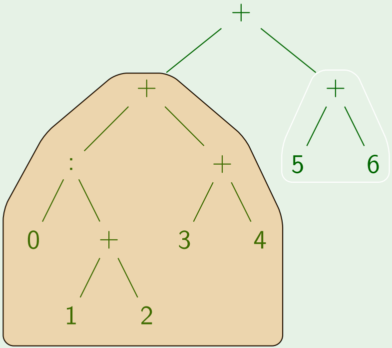


innermost redexes

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

tree representation



leftmost outermost strategy

Example (cont'd)

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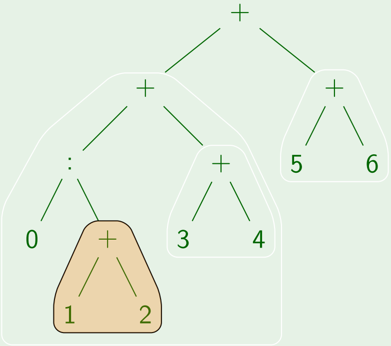

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Example (cont'd)

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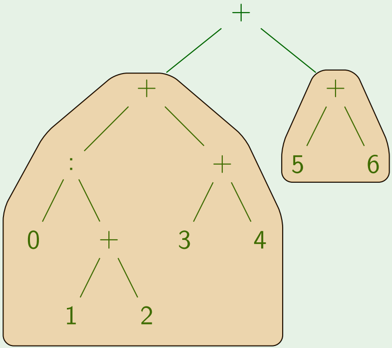

parallel outermost strategy

Example (cont'd)

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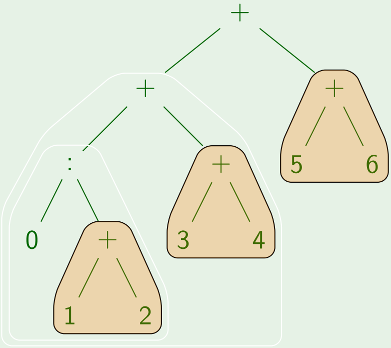

parallel innermost strategy

Definition

Leftmost outermost strategy always reduces the leftmost of the outermost redexes.

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Parallel outermost strategy always reduces all outermost redexes in parallel.

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Definition

The full substitution strategy performs complete development of all redexes.

Theorem

For orthogonal TRSs

- *full substitution and parallel outermost strategies are normalizing*
- *outermost-fair strategies are normalizing*
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$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\mathsf{f}(\mathsf{c}, \mathsf{a})$
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Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- leftmost innermost $\mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- parallel outermost $\mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{\Vert\!\!\!\!\rightarrow} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{b}$
- parallel innermost $\mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{\Vert\!\!\!\!\rightarrow} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \cdots$
- full substitution $\mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{\circ\!\!\!\!\rightarrow} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{\circ\!\!\!\!\rightarrow} \mathsf{b}$