ISR 2026 — Basic Track

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- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Proving strong normalization

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\begin{array}{rcl}
\mathsf{add}(0, y) & \rightarrow & y \\
\mathsf{add}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{add}(x, y))
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Interpretations:

Proving strong normalization

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Interpretations:

> *Find a weight function W from terms to natural numbers in such a way that $W(u) > W(v)$ for all terms u, v satisfying $u \to_R v$.*

Weight functions

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Define:

- $W(0) = 1$
- $W(\mathsf{s}(t)) = W(t) + 1$
- $W(\mathsf{add}(t, u)) = 2W(t) + W(u)$

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Then:

$$\begin{array}{rcl} W(\mathsf{add}(\mathsf{s}(0), 0)) & = & 2 * W(\mathsf{s}(0)) + W(0) = 2 * 2 + 1 = 5 \\ W(\mathsf{s}(\mathsf{add}(0, 0))) & = & W(\mathsf{add}(0, 0)) + 1 = (2 * 1 + 1) + 1 = 4 \\ W(\mathsf{s}(0)) & = & W(0) + 1 = 1 + 1 = 2 \end{array}$$

Problem

There are infinitely many terms!

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Needed:

- $\mathsf{add}(0, 0) \succ 0$
- $\mathsf{add}(0, \mathsf{add}(x, y)) \succ \mathsf{add}(x, y)$
- $\ldots$

Solution

Definition (Reduction Ordering)

A well-founded partial ordering on terms, such that:

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- if $s \succ t$ then $s\sigma \succ t\sigma$ for all substitutions σ
  (we say: $\succ$ is stable)
- if $s \succ t$ then $\mathsf{f}(\dots, s, \dots) \succ \mathsf{f}(\dots, t, \dots)$ for all f
  (we say: $\succ$ is monotonic)

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Let $\mathcal{A}$ be a set and $>$ a well-founded relation on $\mathcal{A}$.

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Here monotonic means:

if for all $a_i, b_i \in \mathcal{A}$ for $i = 1, \ldots, n$ with $a_i > b_i$ for some i and $a_j \geq b_j$ for all $j \neq i$ then

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Examples:

monotonic	*not monotonic*
$\lambda x.\ x$	$\lambda x.\ 2$
$\lambda x.\ x + 1$	$\lambda x, y.\ x + 1$
$\lambda x.\ 2 * x$	$\lambda x, y.\ x * y$
$\lambda x, y.\ x + y$	$\lambda x, y.\ \max(x, y)$
$\lambda x, y.\ 2 * x + y + 1$	

Monotonic algebras (continued)

Define:

- $[\![x]\!] = x$ for a variable
- $[\![\mathsf{f}(s_1, \ldots, s_n))]\!] = [\mathsf{f}]([\![s_1]\!], \ldots, [\![s_n]\!])$

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Theorem

The relation $\succ$ defined by:

$$s \succ t \text{ if and only if } \forall \vec{x}[[\![s]\!] > [\![t]\!]],$$
$$\text{where } \{\vec{x}\} \text{ is the set of variables occurring in } s, t$$

is a reduction ordering if $\ell \succ r$ for all rules $\ell \to r$

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A linear interpretation for (Σ, R) is a function $[\cdot]$ that maps every function symbol f of arity n to a polynomial $[\mathsf{f}]$ of the form:

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with each $a_i > 0$ and $b \geq 0$.

Linear interpretations example

$$\begin{array}{rcl} \mathsf{add}(\mathsf{0}, y) & \to & y \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \end{array}$$

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Choose:

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- $[\mathsf{add}](x, y) = 2 * x + 1 * y + 0$

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Interpretations reduction ordering

Theorem

Given a linear interpretation for (Σ, R).
Define $s \succ t$ *if* $[\![s]\!] > [\![t]\!]$.
If $[\![\ell]\!] > [\![r]\!]$ *for all rules, then* $\succ$ *is a reduction ordering.*

Proof.

A linear interpretation is an instance of a monotonic algebra. ∎

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Practical usage:

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- for all rules:
    - compute $[\![\ell]\!]$ and $[\![r]\!]$
    - show $[\![\ell]\!] > [\![r]\!]$
- conclude termination!

Absolute positiveness

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Observe: $[\![s]\!]$ always has the form $a_1 * y_1 + \cdots + a_m * y_m + b$ with $\{y_1, \ldots, y_m\} = Var(s)$

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Conclude: we must show

$$a_1 * y_1 + \cdots + a_m * y_m + b > c_1 * y_1 + \cdots + c_m * y_m + d$$

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Observe: This holds if and only if:

- each $a_i \geq c_i$
- $b > d$

Class exercise

Prove termination of the following one-rule TRS:

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x)))$$

Finding linear interpretations

How to choose an interpretation function?

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Option 1: guess

Finding linear interpretations

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Option 2: use parametric interpretations and compute the requirements!

Parametric interpretations

$$
\begin{aligned}
[\mathsf{0}] &= \underline{n} \\
[\mathsf{s}] &= \underline{s_0} + \underline{s_1} * x_1 \\
[\mathsf{add}] &= \underline{a_0} + \underline{a_1} * x_1 + \underline{a_2} * x_2
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\begin{aligned}
[\![\mathsf{add}(0, y)]\!] &= \underline{a_0} + \underline{a_1} * \underline{n} + \underline{a_2} * y \\
[\![y]\!] &= y \\
[\![\mathsf{add}(\mathsf{s}(x), y)]\!] &= \underline{a_0} + \underline{a_1} * (\underline{s_0} + \underline{s_1} * x) + \underline{a_2} * y \\
&= \underline{a_0} + \underline{a_1} * \underline{s_0} + \underline{a_1} * \underline{s_1} * x + \underline{a_2} * y \\
[\![\mathsf{s}(\mathsf{add}(x, y))]\!] &= \underline{s_0} + \underline{s_1} * (\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y) \\
&= \underline{s_0} + \underline{s_1} * \underline{a_0} + \underline{s_1} * \underline{a_1} * x + \underline{s_1} * \underline{a_2} * y
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Parametric interpretations (continued)

We must show that:

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That is:

$$(\underline{a_0} + \underline{a_1} * \underline{n}) + \underline{a_2} * y > y$$

and

$$(\underline{a_0} + \underline{a_1} * \underline{s_0}) + (\underline{a_1} * \underline{s_1}) * x + \underline{a_2} * y > (\underline{s_0} + \underline{s_1} * \underline{a_0}) + (\underline{s_1} * \underline{a_1}) * x + (\underline{s_1} * \underline{a_2}) * y$$

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Using absolute positiveness, this holds exactly if:

$$\begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{n} & > & 0 \\ \\ \underline{a_2} & \geq & 1 \end{array} \qquad \begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{s_0} & > & \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} & \geq & \underline{s_1} * \underline{a_1} \\ \underline{a_2} & \geq & \underline{s_1} * \underline{a_2} \end{array}$$

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Parametric interpretations (continued)

That is:

$$(\underline{a_0} + \underline{a_1} * \underline{n}) + \underline{a_2} * y > y$$

and

$$(\underline{a_0} + \underline{a_1} * \underline{s_0}) + (\underline{a_1} * \underline{s_1}) * x + \underline{a_2} * y > (\underline{s_0} + \underline{s_1} * \underline{a_0}) + (\underline{s_1} * \underline{a_1}) * x + (\underline{s_1} * \underline{a_2}) * y$$

Using absolute positiveness, this holds exactly if:

$$\begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{n} & > & 0 \\ \underline{a_2} & \geq & 1 \\ \underline{s_1} & \geq & 1 \end{array} \qquad \begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{s_0} & > & \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} & \geq & \underline{s_1} * \underline{a_1} \\ \underline{a_2} & \geq & \underline{s_1} * \underline{a_2} \\ \underline{a_1} & \geq & 1 \end{array}$$

Practical observation: usually values in $\{0, 1, 2, 3\}$ suffice!

Linear interpretations are insufficient!

$$\begin{array}{rcl} \mathsf{add}(0, y) & \rightarrow & y \\ \mathsf{add}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{add}(x, y)) \\ \mathsf{mul}(0, y) & \rightarrow & 0 \\ \mathsf{mul}(\mathsf{s}(x), y) & \rightarrow & \mathsf{add}(y, \mathsf{mul}(x, y)) \end{array}$$

Obvious extension: polynomial interpretations

$$\begin{array}{rcl} \mathsf{add}(0, y) & \to & y \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \\ \mathsf{mul}(0, y) & \to & 0 \\ \mathsf{mul}(\mathsf{s}(x), y) & \to & \mathsf{add}(y, \mathsf{mul}(x, y)) \end{array}$$

Interpretation:

$$\begin{array}{rcl} [0] & = & 0 \\ [\mathsf{s}](x) & = & x + 1 \\ [\mathsf{add}](x, y) & = & 2x + y + 1 \\ [\mathsf{mul}](x, y) & = & 2x + y + 2xy + 1 \end{array}$$

Definition (Polynomial interpretation)

Assume given a TRS with rules R and signature Σ.

A polynomial interpretation for (Σ, R) is a function $[\cdot]$ that maps every function symbol f of arity n to a monotonic polynomial $[\mathsf{f}]$.

Absolute positiveness for polynomial interpretations

Same idea:

$$ax_1 + bx_2 + cx_1x_2 + d > ex_1 + fx_3 + gx_2^2 + h$$

if:

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- $(b \geq 0)$
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- Why polynomial? Why not allow min, max, $\mathtt{exp}$, ...?
- Why natural numbers? Why not $\mathbb{N} \setminus \{0\}$? Or ordinals? Or ...?

Example: polynomial interpretations in $\mathbb{N} \setminus \{0\}$

$$
\begin{array}{rcl}
\mathsf{add}(\mathsf{0}, y) & \to & y \\
\mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \\
\mathsf{mul}(\mathsf{0}, y) & \to & \mathsf{0} \\
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\end{array}
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Example: marked integers

$$\mathsf{a}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(\mathsf{a}(x)))$$

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$$a(a(x)) \to a(b(a(x)))$$

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$$\begin{array}{rcl} [a](x^{a}) & = & (x+1)^{a} \\ [a](x^{b}) & = & x^{a} \\ [b](x^{?}) & = & x^{b} \end{array}$$

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Where: $x^c > y^d$ if $x > y$ and $c = d$

Example: a terminating TRS!

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The following monotonic algebra works:

- $(\mathcal{A}, \succ) = (\mathcal{T}(\Sigma, \mathcal{X}), \rightarrow_{\mathcal{R}})$
- $[\mathsf{f}](t_1, \ldots, t_n) = \mathsf{f}(t_1, \ldots, t_n)$

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- $(\mathcal{A}, \succ) = (\mathcal{T}(\Sigma, \mathcal{X}), \to_{\mathcal{R}})$
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Corollary

A TRS is terminating if and only if its rules are oriented by a monotonic algebra.

Difficult example

$$
\begin{array}{rcl}
\mathsf{minus}(x, \mathsf{0}) & \rightarrow & x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \rightarrow & \mathsf{minus}(x, y) \\
\mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \rightarrow & \mathsf{0} \\
\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \rightarrow & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y)))
\end{array}
$$

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The good:

- intuitive: a term is bigger than its subterms
- useful property in some proofs (e.g., soundness of *superposition*)

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A reduction ordering $\succ$ has the subterm property if $\mathsf{f}(\ldots, s, \ldots) \succeq s$ for all f.

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The good:

- intuitive: a term is bigger than its subterms
- useful property in some proofs (e.g., soundness of *superposition*)

The bad:

- Not all TRSs can be ordered this way!

Limitations of the subterm property

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\begin{array}{rcl}
\mathsf{minus}(x, \mathsf{0}) & \to & x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
\mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \to & \mathsf{0} \\
\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y)))
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\end{array}
$$

If $\mathsf{minus}(x, y) \succ y$ and $\mathsf{s}(a) \succ a$, then:

$$
\mathsf{s}(\mathsf{quot}(\underline{\mathsf{minus}(x, y)}, \mathsf{s}(y))) \quad \succ \quad \mathsf{s}(\mathsf{quot}(\underline{y}, \mathsf{s}(y)))
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Hence, if the last rule is oriented with $\succ$, then by stability:

$$\begin{array}{rcl} \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(\underline{\mathsf{s}(x)})) & \succ & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, \underline{\mathsf{s}(x)}), \mathsf{s}(\underline{\mathsf{s}(x)}))) \\ & \succ & \mathsf{quot}(\underline{\mathsf{s}(x)}, \mathsf{s}(\underline{\mathsf{s}(x)})) \end{array}$$

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Hence, this system cannot be ordered using an interpretation to $\mathbb{N}$.

Mapping to a pair

Choice:

- $\mathcal{A} = \mathbb{N}^2$
- $(a, b) \succ (c, d)$ if $a > c$ and $b \geq d$

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Intuition: first argument bounds cost, second bounds size

Example: cost-size interpretation

$$
\begin{array}{rcl}
\mathsf{minus}(x, \mathsf{0}) & \to & x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
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\end{array}
$$

$$
\begin{array}{rcllll}
[\mathsf{0}] & = & \langle & 0 & , & 0 \rangle \\
[\mathsf{s}](\langle c, s \rangle) & = & \langle & c & , & s + 1 \rangle \\
[\mathsf{minus}](\langle c_1, s_1 \rangle, \langle c_2, s_2 \rangle) & = & \langle & c_1 + c_2 + s_2 + 1 & , & s_1 \rangle \\
[\mathsf{quot}](\langle c_1, s_1 \rangle, \langle c_2, s_2 \rangle) & = & \langle & c_1 + c_2 + s_1(1 + s_2 + c_2) + 1 & , & s_1 \rangle
\end{array}
$$

Example: using subtraction

$$\mathsf{f}(\mathsf{s}(x)) \to \mathsf{f}(\mathsf{p}(\mathsf{s}(x))) \qquad \mathsf{p}(0) \to 0 \qquad \mathsf{p}(\mathsf{s}(x)) \to x$$

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$$\mathsf{f}(\mathsf{s}(x)) \to \mathsf{f}(\mathsf{p}(\mathsf{s}(x))) \qquad \mathsf{p}(\mathsf{0}) \to \mathsf{0} \qquad \mathsf{p}(\mathsf{s}(x)) \to x$$

$$\begin{array}{rcllcll}
[\mathsf{0}] & = & \langle & 0 & , & 0 & \rangle \\
[\mathsf{s}](\langle x_1, x_2 \rangle) & = & \langle & x_1 & , & x_2 + 1 & \rangle \\
[\mathsf{p}](\langle x_1, x_2 \rangle) & = & \langle & x_1 + 1 & , & \max(x_2 - 1, 0) & \rangle \\
[\mathsf{f}](\langle x_1, x_2 \rangle) & = & \langle & x_1 + 2x_2 & , & 0 & \rangle
\end{array}$$

Matrix interpretations

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(x)))$$

$$\begin{array}{lclclcll} [\mathsf{b}](\langle c, s\rangle) & = & \langle & c & , & s+1 & \rangle \\ [\mathsf{a}](\langle c, s\rangle) & = & \langle & c+s & , & 2s & \rangle \end{array}$$

Matrix interpretations

$$a(b(x)) \to b(b(a(x)))$$

$$\begin{array}{rcllll} [b](\langle c, s\rangle) & = & \langle & c & , & s+1 & \rangle \\ [a](\langle c, s\rangle) & = & \langle & c+s & , & 2s & \rangle \end{array}$$

$$[b] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } [a] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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b:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c \\ s \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ s+1 \\ 1 \end{pmatrix}$$

a:

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