ISR 2026 — Basic Track

Jörg Endrullis
Cynthia Kop
Femke van Raamsdonk

partially based on slides by: Aart Middeldorp

hosting institution:
Radboud Universiteit Nijmegen
The Netherlands

- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Historic Overview

Historical overview: LPO and RPO

Plaisted [1978] and Dershowitz [1979]:

- **recursive path order (RPO)**

Kamin and Lévy [1980]:

- **lexicographic path order (LPO)**

Bergsta and Klop [1985]:

- iterative version of RPO, defined via rewriting

Buchholz [1995]:

- **simplified proof**: Kruskal not needed

Klop, van Oostrom and de Vrijer [2005]:

- **iterative lexicographic path order (ILPO)**

Lexicographic Path Order (LPO)

Lexicographic Path Order (LPO)

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s = f(s_1, \ldots, s_m) \succ_{lpo} t$ iff

(LPO1) $s_i = t$ or $s_i \succ_{lpo} t$ for some $1 \leq i \leq m$, or

(LPO2) $t = g(t_1, \ldots, t_n)$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and

- (LPO2a) $f \succ g$, or
- (LPO2b) $f = g$, and $\exists i$: $s_1 = t_1$, …, $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

Iterative Lexicographic Path Order (ILPO)

ILPO... The star TRS $\mathcal{L}\mathsf{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathsf{ex}_{\succ}$

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- Signature: $\Sigma \uplus \Sigma^*$, where $\Sigma^* = \{f^* \mid f \in \Sigma\}$

  f^* is fresh and has the same arity as f

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$$f(\vec{x}) \rightarrow_{\mathrm{put}} f^*(\vec{x})$$

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$$f(\vec{x}) \rightarrow_{\mathrm{put}} f^*(\vec{x})$$
$$f^*(\vec{x}) \rightarrow_{\mathrm{select}} x_i$$

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- Reduction rules (four types, for arbitrary $f, g \in \Sigma$):

$$
\begin{aligned}
f(\vec{x}) &\to_{\mathrm{put}} f^*(\vec{x}) \\
f^*(\vec{x}) &\to_{\mathrm{select}} x_i \\
f^*(\vec{x}) &\to_{\mathrm{copy}} g(f^*(\vec{x}), \ldots, f^*(\vec{x})) \qquad \text{if } f \succ g
\end{aligned}
$$

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f(\vec{x}) &\rightarrow_{\mathrm{put}} f^*(\vec{x}) && \\
f^*(\vec{x}) &\rightarrow_{\mathrm{select}} x_i && \\
f^*(\vec{x}) &\rightarrow_{\mathrm{copy}} g(f^*(\vec{x}), \ldots, f^*(\vec{x})) && \text{if } f \succ g \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) &\rightarrow_{\mathrm{lex}} f(\vec{x}, g^*(\vec{y}), l, \ldots, l) && \text{where } l = f^*(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
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\end{aligned}
$$

Definition (ILPO)

$\succ_{ilpo}$ is the restriction of $\to^+_{\mathcal{L}\mathrm{ex}_{\succ}}$ to terms over Σ, i.e.

$$t \succ_{ilpo} s \iff t \to^+_{\mathcal{L}\mathrm{ex}_{\succ}} s \;\wedge\; t, s \in T(\Sigma \uplus V)$$

ILPO

Claim

$\succ_{ilpo}$ is a reduction order, that is:

- $\succ_{ilpo}$ is well-founded, and
- closed under substitution and contexts

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The closure under substitutions and contexts is immediate since $\to^{+}_{\mathcal{L}\mathsf{ex}_{\succ}}$ is.

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Only required: proof of termination of $\succ_{ilpo}$.

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Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

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The closure under substitutions and contexts is immediate since $\rightarrow^{+}_{\mathcal{L}\mathsf{ex}_{\succ}}$ is.

Only required: proof of termination of $\succ_{ilpo}$.

Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

Proof. $\succ_{ilpo}$ is a reduction order with $\mathcal{R} \subseteq \succ_{ilpo}$.

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\to x \\
\mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

Example, Addition and multiplication

$$\boxed{\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

Example, Addition and multiplication

$$
\begin{array}{rcl}
\mathtt{A}(x, \mathtt{0}) & \rightarrow & x \\
\mathtt{A}(x, \mathtt{S}(y)) & \rightarrow & \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) & \rightarrow & \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) & \rightarrow & \mathtt{A}(x, \mathtt{M}(x, y))
\end{array}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0})$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0})$$

Example, Addition and multiplication

$$\boxed{\begin{aligned} \mathtt{A}(x, 0) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, 0) &\to 0 \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, 0) \to_{\mathsf{put}} \mathtt{A}^*(x, 0) \to_{\mathsf{select}} x$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x,\mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x$$

$$\mathtt{A}(x,\mathtt{S}(y))$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \end{aligned}$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x,\mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x$$

$$\mathtt{A}(x,\mathtt{S}(y)) \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y)))$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to x \\
\mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y)) \\
\mathtt{M}(x,\mathtt{0}) &\to \mathtt{0} \\
\mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\
\mathtt{A}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\
&\to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y)))
\end{aligned}
$$

Example, Addition and multiplication

$$\boxed{\begin{array}{rcl} \mathtt{A}(x, \mathtt{0}) & \to & x \\ \mathtt{A}(x, \mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) & \to & \mathtt{A}(x, \mathtt{M}(x, y)) \end{array}}$$

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Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\rightarrow x \\ \mathtt{A}(x,\mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\rightarrow \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \rightarrow_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \end{aligned}$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \end{aligned}$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\mathsf{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \end{aligned}$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to x\\
\mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y))\\
\mathtt{M}(x,\mathtt{0}) &\to \mathtt{0}\\
\mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x\\
\mathtt{A}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y)))\\
&\to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\mathsf{select}} \mathtt{S}(\mathtt{A}(x,y))\\
\mathtt{M}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\mathsf{select}} \mathtt{0}\\
\mathtt{M}(x,\mathtt{S}(y))
\end{aligned}
$$

Example, Addition and multiplication

$$\boxed{\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\mathsf{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \end{aligned}$$

Example, Addition and multiplication

$$\boxed{\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{array}{rl} \mathtt{A}(x,\mathtt{0}) & \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ & \to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\mathsf{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y))) \end{array}$$

Example, Addition and multiplication

$$\boxed{\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{array}{rl} \mathtt{A}(x,\mathtt{0}) & \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\mathsf{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ & \to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\mathsf{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y))) \\ & \to_{\mathsf{select}} \mathtt{A}(x,\mathtt{M}^*(x,\mathtt{S}(y))) \end{array}$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\text{select}} x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\ &\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x, \mathtt{0}) \to_{\text{select}} \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \to_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y))) \end{aligned}$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \rightarrow & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \rightarrow & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \rightarrow & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \rightarrow & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

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Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathrm{ex}^{\omega}$

But $\mathcal{L}$ex is in general not terminating, e.g. if $\mathtt{A} > \mathtt{S}$, then

$$\begin{aligned}
\mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^*(x,y) \\
&\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\
&\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\
&\dots
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Starred symbol $\mathtt{A}^*$ is ‘used’ infinitely often.

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&\dots
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This is essential in any infinite reduction!

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Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathrm{ex}^{\omega}$

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&\dots
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Starred symbol $\mathtt{A}^*$ is 'used' infinitely often.

This is essential in any infinite reduction!

Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

$\Rightarrow$ Yields a terminating TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$.

Auxiliary TRS with numerical control symbols

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^{\omega}$, where $\Sigma^{\omega} = \{f^n \mid f \in \Sigma,\ n \in \mathbb{N}\}$

  f^n is fresh and has same arity as f

- Reduction rules:

$$
\begin{aligned}
f(\vec{x}) &\rightarrow_{\mathsf{put}} f^n(\vec{x}) \\
f^n(\vec{x}) &\rightarrow_{\mathsf{select}} x_i \\
f^{n+1}(\vec{x}) &\rightarrow_{\mathsf{copy}} g(f^n(\vec{x}), \ldots, f^n(\vec{x})) && \text{if } f \succ g \\
f^{n+1}(\vec{x}, g(\vec{y}), \vec{z}) &\rightarrow_{\mathsf{lex}} f(\vec{x}, g^n(\vec{y}), l, \ldots, l) && \text{where } l = f^n(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

Back and forth between $\mathcal{L}$ex and $\mathcal{L}\text{ex}^\omega$

From $\rightarrow_{\mathcal{L}\text{ex}^\omega}$ to $\rightarrow_{\mathcal{L}\text{ex}}$:

- every reduction can be transformed by replacing f^n by f^*.

Back and forth between $\mathcal{L}\mathsf{ex}$ and $\mathcal{L}\mathsf{ex}^\omega$

From $\rightarrow_{\mathcal{L}\mathsf{ex}^\omega}$ to $\rightarrow_{\mathcal{L}\mathsf{ex}}$:

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From $\rightarrow_{\mathcal{L}\mathsf{ex}}$ to $\rightarrow_{\mathcal{L}\mathsf{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

Back and forth between $\mathcal{L}$ex and $\mathcal{L}\text{ex}^\omega$

From $\to_{\mathcal{L}\text{ex}^\omega}$ to $\to_{\mathcal{L}\text{ex}}$:

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From $\to_{\mathcal{L}\text{ex}}$ to $\to_{\mathcal{L}\text{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

For example:

$$\begin{aligned} \mathtt{M}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \to_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

becomes:

$$\begin{aligned} \mathtt{M}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^2(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^1(x, \mathtt{S}(y)), \mathtt{M}^1(x, \mathtt{S}(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}^1(x, \mathtt{S}(y))) \to_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^0(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

$\mathcal{L}\mathsf{ex}$ and $\mathcal{L}\mathsf{ex}^\omega$

Theorem

$\to^+_{\mathcal{L}\mathsf{ex}_\succ}$ *and* $\to^+_{\mathcal{L}\mathsf{ex}^\omega_\succ}$ *coincide on* $T(\Sigma \uplus V)$

$\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

Theorem

$\to^+_{\mathcal{L}\mathrm{ex}_\succ}$ and $\to^+_{\mathcal{L}\mathrm{ex}^\omega_\succ}$ coincide on $T(\Sigma \uplus V)$

Note that the infinite reduction

$$\begin{aligned} \mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^*(x,y) \\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\ &\dots \end{aligned}$$

cannot be lifted:

$$\begin{aligned} \mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^?(x,y) \\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^{?-1}(x,y)) \\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^{?-2}(x,y))) \\ &\dots \end{aligned}$$

Termination of $\mathcal{L}\mathsf{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n, > \rangle$.

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- Here $(\rightarrow_{\mathcal{L}\mathsf{ex}^{\omega}_{\succ}})^n$ is the lexicographic order on n-tuples

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- Here $(\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

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- No label ℓ counts as ∞ with $\infty > n$.

In general a term is SN if all one-step reducts are SN.

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- Here $(\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

In general a term is SN if all one-step reducts are SN.

$\Rightarrow$ We check all one step reducts of $f^{\ell}(t_1, \dots, t_n)$.

Termination of $\mathcal{L}\mathsf{ex}^\omega$ à la Buchholz

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Case 1. Internal step $f^{\ell}(\ldots, t_i, \ldots) \to f^{\ell}(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \dots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \dots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n, > \rangle$.

Case 1. Internal step $f^{\ell}(\dots, t_i, \dots) \rightarrow f^{\ell}(\dots, t_i', \dots)$.
The triple decreases in the second component.

Case 2. $f(t_1, \dots, t_n) \rightarrow_{\mathsf{put}} f^{n}(t_1, \dots, t_n)$.
We have a decrease in the third component.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

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We have a decrease in the third component.

Case 3. $f^{n}(t_1, \ldots, t_n) \rightarrow_{\mathsf{select}} t_i$.
By assumption t_i is SN.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n, > \rangle$.

Case 4. $f^{n+1}(\vec{t}) \to_{\mathsf{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.
By IH the arguments $f^n(\vec{t})$ of g are SN since $n + 1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Termination of $\mathcal{L}\mathrm{ex}^\omega$ à la Buchholz

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$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathrm{ex}^\omega_\succ})^n, > \rangle$.

Case 4. $f^{n+1}(\vec{t}) \to_{\mathrm{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.
By IH the arguments $f^n(\vec{t})$ of g are SN since $n + 1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Case 5. $f^{n+1}(\vec{t}, g(\vec{s}), \vec{r}) \to_{\mathrm{lex}} f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$, $l = f^n(\vec{t}, g(\vec{y}), \vec{r})$.
By assumption the arguments $\vec{t}$ and $g(\vec{s})$ are SN.
By IH we get l is SN.
Since $g(\vec{s}) \to_{\mathrm{put}} g^n(\vec{s})$ we get

- $g^n(\vec{s})$ is SN, and
- the triple decreases in the second component.

Thus by IH $f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$ is SN.

Termination of $\mathcal{L}\mathsf{ex}$ and $\mathcal{L}\mathsf{ex}^\omega$

Hence we have proven:

Theorem

$\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ}$ *is terminating*

Termination of $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^\omega$

Hence we have proven:

Theorem

$\to_{\mathcal{L}\mathrm{ex}^\omega_\succ}$ *is terminating*

Corollary

$\to^+_{\mathcal{L}\mathrm{ex}}$ *is terminating on* $T(\Sigma \uplus V)$

Proof. $\to^+_{\mathcal{L}\mathrm{ex}}$ and $\to^+_{\mathcal{L}\mathrm{ex}^\omega}$ coincide on $T(\Sigma \uplus V)$

Example, Ackermann Function

Example

$$\begin{aligned}
\mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

Find an order $\succ$ on Σ which proves termination.

Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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We get the following derivations:

$$\mathrm{Ack}(0, y)$$

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$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y})$$

Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y}))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

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We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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$$\mathrm{Ack}(\mathrm{s}(x), 0)$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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Find an order $\succ$ on Σ which proves termination.

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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$

Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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Find an order $\succ$ on Σ which proves termination.

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We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$

Example, Ackermann Function

Example

$$\begin{aligned}
\mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\
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We get the following derivations:

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\mathrm{Ack}(0, y) &\to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y}) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \to_{\mathsf{copy}} \mathrm{Ack}(\mathrm{x}, \mathrm{s}(\mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)))
\end{aligned}$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\begin{aligned} \mathrm{Ack}(\mathrm{s}(x), 0) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \\ &\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \to_{\mathsf{copy}} \mathrm{Ack}(\mathrm{x}, \mathrm{s}(\mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))) \\ &\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{s}(0)) \end{aligned}$$

Example, Ackermann Function

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Example, Ackermann Function

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$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

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$$\begin{aligned}\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\ &\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})))\end{aligned}$$

Example, Ackermann Function

Example

$$\begin{aligned}
\mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

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\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})))
\end{aligned}$$

Example, Ackermann Function

Example

$$\begin{aligned} \mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y)) \end{aligned}$$

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Example, Ackermann Function

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&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}^*(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

Hence we have proven termination.

Lexicographic Path Order (LPO)

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s = f(s_1, \ldots, s_m) \succ_{lpo} t$ iff

(LPO1) $s_i = t$ or $s_i \succ_{lpo} t$ for some $1 \leq i \leq m$, or

(LPO2) $t = g(t_1, \ldots, t_n)$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and

- (LPO2a) $f \succ g$, or
- (LPO2b) $f = g$, and $\exists i$: $s_1 = t_1$, …, $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

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(LPO2b) $f = g$, and $\exists i$: $s_1 = t_1$, ..., $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

Theorem

$\succ_{ilpo}$ is equivalent with $\succ_{lpo}$

Recursive Path Order (RPO)

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Multiset: A multiset over A is a function $M : A \to \mathbb{N}$.

Multiset extension: For a strict order $\succ$, define $\succ^{mul}$ on finite multisets by:

$$X \uplus M \succ^{mul} Y \uplus M \quad \Longleftrightarrow \quad X \neq \emptyset, \text{ and } \forall\, y \in Y\ \exists\, x \in X :\ x \succ y$$

$\{\, 2, 4, 4, 7 \,\} \succ^{mul} \{1, 2, 2, 2, 3, 4, 7\}$

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- (RPO2a) $f \succ g$, or
- (RPO2b) $f = g$, and $\{s_1, \ldots, s_m\} \succ_{rpo}^{mul} \{t_1, \ldots, t_n\}$.

Extensions of LPO and RPO

Extensions of LPO and RPO

Status: per symbol choice multiset or lexicographically

Jouannaud, Lescanne & Reinig [1982]; Lescanne [1990]

Semantic path orderings: replace the precedence by a quasi-order on terms

Kamin & Lévy [1980]; Dershowitz & Hoot [1995]

Yamada, Kusakari & Sakabe [2015], Saito & Hirokawa [2023]

reduction order capturing dependency pairs

Borralleras, Ferreira & Rubio [2000]

Modulo AC: compatible with associativity and commutativity

Bachmair & Plaisted [1985], Rubio & Nieuwenhuis [1995]

Kapur & Sivakumar [1997], Rubio [2002], Borralleras & Rubio [2003]

Higher-order: lifted to simply-typed λ-terms: higher-order RPO

Jouannaud & Rubio [1999, 2007]

computability path ordering

Blanqui, Jouannaud & Rubio [2015]