ISR 2026 — Basic Track

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- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Weak Church Rosser or local confluence

two divergent steps

- caused by parallel redexes are joinable
- caused by nested non-overlapping redexes are joinable
- two divergent steps caused by overlapping redexes are joinable
  if the underlying critical pair(s) is (are) joinable

critical pairs

- arise from a most general common instance of two left-hand sides
- can be computed using a unification algorithm
- are ordered
- from an overlap at the root by different rules we get two

for terminating TRSs, WCR and hence CR are decidable

Outline

Example (equational system for addition)

$$0 + y \approx y$$

$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

what follows from these equations?

what does it mean to ‘follow’ from equations?

Example (equational system for booleans)

$$
\begin{aligned}
\mathsf{and}(\mathsf{true}, x) &\approx x \\
\mathsf{and}(\mathsf{false}, x) &\approx \mathsf{false} \\
\mathsf{not}(\mathsf{true}) &\approx \mathsf{false} \\
\mathsf{not}(\mathsf{false}) &\approx \mathsf{true} \\
\mathsf{or}(x, y) &\approx \mathsf{not}(\mathsf{and}(\mathsf{not}(x), \mathsf{not}(y)))
\end{aligned}
$$

Example (equational systems for groups)

$$\begin{aligned} \mathsf{e} \cdot x &\approx x \\ x^{-} \cdot x &\approx \mathsf{e} \\ (x \cdot y) \cdot z &\approx x \cdot (y \cdot z) \end{aligned}$$

Definition

An equational system (ES) is pair $(\Sigma, \mathcal{E})$ consisting of

- Σ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\Sigma, \mathcal{X})$

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Example (addition again)

ES $(\Sigma, \mathcal{E})$ with signature Σ

$$0 \text{ (constant)} \qquad \mathsf{s} \text{ (unary)} \qquad + \text{ (binary, infix)}$$

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

often we only give the equations

What follows from an equation?

Intuitively: what we get ‘replacing equals by equals’.

Informal example:

$$\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

This can be made more formal.

Equality is reflexive, symmetric, and transitive.

We can use a substitution instance of a given equation.

We can use an equation in a larger term.

Inference Rules for Derivability

[r] reflexivity $\dfrac{}{t \approx t}$ $\forall\ t$

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[r]	reflexivity	$\dfrac{}{t \approx t}$	$\forall\ t$
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[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$

Inference Rules for Derivability

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[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$
[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n)}$	$\forall$ n-ary f

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Example

ES $\mathcal{E}$

$$
\begin{aligned}
0 + y &\approx y \\
\mathsf{s}(x) + y &\approx \mathsf{s}(x + y)
\end{aligned}
$$

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$$\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}\,[\mathtt{a}] \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[\mathtt{a}]}{\mathsf{s}(0 + s(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathtt{c}]}{\dfrac{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[\mathtt{c}]}\,[\mathtt{t}]$$

Definition (Syntactic Consequence)

$\mathcal{E} \vdash s \approx t \quad (s \approx_{\mathcal{E}} t)$ if equation $s \approx t$ is derivable.

So: if there exists a derivation that has our equality as conclusion.

We have seen: $\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

It may be hard to show that an equation is not derivable.

a more attractive approach

instead of making derivations in the formal system,

we use a TRS $\mathcal{E}$

with rules $l \to r$ and $r \to l$ for every equation $l \approx r$ in $\mathcal{E}$

example:

$$\begin{aligned} 0 + y &\to y \\ y &\to 0 + y \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\ \mathsf{s}(x + y) &\to \mathsf{s}(x) + y \end{aligned}$$

a more attractive approach

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example:

$$\begin{aligned} 0 + y &\to y \\ y &\to 0 + y \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\ \mathsf{s}(x + y) &\to \mathsf{s}(x) + y \end{aligned}$$

Theorem

$\forall$ *ES* $\mathcal{E}$ $\quad \mathcal{E} \vdash s \approx t \quad \Longleftrightarrow \quad s \leftrightarrow^*_{\mathcal{E}} t$

Outline

Given a signature Σ

a Σ-algebra is given by:

a non-empty carrier set A

for every function symbol f of arity n a mapping

$[\mathrm{f}] : A^n \to A$

Examples (algebras for addition)

- $\mathcal{A}$: carrier set is $\mathbb{N}$ and the interpretation of the function symbols is
  $[0] = 0, \quad [s](x) = x + 1, \quad [+](x, y) = x + y$
- $\mathcal{B}$: carrier set is $\mathbb{N}$ and the interpretation of the function symbols is
  $[0] = 1, \quad [s](x) = x + 1, \quad [+](x, y) = 2x + y$
- $\mathcal{C}$: carrier set is $A = \{a, b, c\}$ and the interpretation of the function symbols is $[0] = a$, and

[s]	
a	a
b	a
c	a

[+]	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

Definition

An equation $s \approx t$ is valid in the Σ-algebra $\mathcal{A}$ $\quad (\mathcal{A} \vDash s \approx t)$ if

$$[s, \alpha]_{\mathcal{A}} = [t, \alpha]_{\mathcal{A}}$$

for all assignments α.

assignments map variables to elements in the carrier set

we do not always mention them explicitly

Back to the examples

the equation $\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$ is valid in $\mathcal{A}$

the equation $\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$ is not valid in $\mathcal{B}$

the equation $\mathsf{s}(\mathsf{s}(0)) \approx 0$ is valid in $\mathcal{C}$

Definition

Σ-algebra $\mathcal{A}$ is **model** of ES $(\Sigma, \mathcal{E})$

if $\mathcal{A} \vDash s \approx t$ for all equations $s \approx t \in \mathcal{E}$.

Examples

- $\mathcal{A}$ is a model for the equational system for addition
- $\mathcal{B}$ is not a model for the equational system for addition
- $\mathcal{C}$ is not a model for the equational system for addition:

  $0 + y \approx y$ is not valid in $\mathcal{C}$ (taking a for y)

Definition

- $\mathcal{E} \vDash s \approx t$ if equation $s \approx t$ is valid in all models of $\mathcal{E}$.
- The equational theory of $\mathcal{E}$ consists of all equations $s \approx t$ such that $\mathcal{E} \vDash s \approx t$.

Example

$$\mathcal{E} \not\models x + y \approx y + x$$

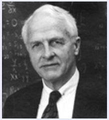
Consider model $\mathcal{D} = (\mathbb{N}, [\cdot])$

with $\quad [0] = 0, \quad [s](x) = x, \quad [+](x, y) = y.$

Theorem (Birkhoff)

Equational reasoning is sound and complete

$\forall$ *ES* $\mathcal{E}$: $\quad \mathcal{E} \vdash s \approx t \quad \Longleftrightarrow \quad \mathcal{E} \vDash s \approx t$



Example

$\mathcal{E} \vDash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$ because $\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$\mathcal{E} \nvdash x + y \approx y + x$ because $\mathcal{E} \nvDash x + y \approx y + x$

if we want to show that an equation is not derivable,

we can give a model in which that equation is not valid

$\mathcal{E} \vdash s \approx t$ if there exists a derivation with conclusion $s \approx t$

$\mathcal{E} \vDash s \approx t$ if $a \approx t$ is valid in all models of $\mathcal{E}$

Outline

Validity Problem

instance: ES $(\Sigma, \mathcal{E})$ terms $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$

question: $\mathcal{E} \vDash s \approx t$? (or equivalently $\mathcal{E} \vdash s \approx t$?)

Validity Problem

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Theorem

The validity problem is undecidable.

Validity Problem

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Validity Problem

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question: $\mathcal{E} \vDash s \approx t$? (or equivalently $\mathcal{E} \vdash s \approx t$?)

Theorem

The validity problem is undecidable.

Example (Combinatory Logic)

$$\begin{aligned} \mathsf{I} \cdot x &\approx x \\ (\mathsf{K} \cdot x) \cdot y &\approx x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\approx (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \vDash s \approx t$?

Theorem

The validity problem is *undecidable*

here $\rightarrow_{\mathcal{E}}$ is the TRS that has as rules $l \rightarrow r$ and $r \rightarrow r$ for every equation $l \approx r$ in $\mathcal{E}$

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \vDash s \approx t$?

Theorem

The validity problem is decidable for ES $\mathcal{E}$ if $\exists$ finite TRS R such that

1. *R is complete (confluent and terminating)*

here $\rightarrow_{\mathcal{E}}$ is the TRS that has as rules $l \rightarrow r$ and $r \rightarrow r$ for every equation $l \approx r$ in $\mathcal{E}$

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \vDash s \approx t$?

Theorem

The validity problem is decidable *for ES $\mathcal{E}$ if $\exists$ finite TRS R such that*

1. *R is complete (confluent and terminating)*
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

here $\to_{\mathcal{E}}$ is the TRS that has as rules $l \to r$ and $r \to r$ for every equation $l \approx r$ in $\mathcal{E}$

Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^-$ (unary, postfix) $\quad \cdot$ (binary, infix)

ES $\quad \mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\quad \mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

TRS $\qquad R$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
x^- \cdot x & \to & \mathsf{e} & x \cdot x^- & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^- & \to & \mathsf{e} & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & x \cdot (x^- \cdot y) & \to & y
\end{array}
$$

Example (Group Theory)

signature	e (constant) $\quad {}^-$ (unary, postfix) $\quad \cdot$ (binary, infix)		
ES	$\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$
theorems	$\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$		
TRS	$\mathsf{e} \cdot x \to x$	$x \cdot \mathsf{e} \to x$	R
	$x^- \cdot x \to \mathsf{e}$	$x \cdot x^- \to \mathsf{e}$	
	$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$	$x^{--} \to x$	
	$\mathsf{e}^- \to \mathsf{e}$	$(x \cdot y)^- \to y^- \cdot x^-$	
	$x^- \cdot (x \cdot y) \to y$	$x \cdot (x^- \cdot y) \to y$	

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \downarrow_R \mathsf{e} \qquad (x \cdot y)^- \downarrow_R y^- \cdot z^-$

TRS R

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \implies \mathcal{E}$ has decidable validity problem

Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^-$ (unary, postfix) $\quad \cdot$ (binary, infix)

ES $\quad \mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\quad \mathsf{e}^- \downarrow_R \mathsf{e} \qquad (x \cdot y)^- \downarrow_R y^- \cdot z^-$

TRS

$$
\begin{array}{rlrl}
\mathsf{e} \cdot x & \to x & x \cdot \mathsf{e} & \to x \\
x^- \cdot x & \to \mathsf{e} & x \cdot x^- & \to \mathsf{e} \\
(x \cdot y) \cdot z & \to x \cdot (y \cdot z) & x^{--} & \to x \\
\mathsf{e}^- & \to \mathsf{e} & (x \cdot y)^- & \to y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to y & x \cdot (x^- \cdot y) & \to y
\end{array}
\qquad R
$$

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \implies \mathcal{E}$ has decidable validity problem
- How to compute R ?

Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

ES $\quad \mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\quad \mathsf{e}^{-} \downarrow_R \mathsf{e} \qquad (x \cdot y)^{-} \downarrow_R y^{-} \cdot z^{-}$

TRS $\qquad R$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
x^{-} \cdot x & \to & \mathsf{e} & x \cdot x^{-} & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\
x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y
\end{array}
$$

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem
- How to compute R ? $\qquad$ completion

Outline

completion: intuition

if a terminating TRS is not WCR

add a rule to make it WCR

preserving termination

this works often

Example

TRS R

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

Example

TRS R

$$
\begin{array}{llcl@{\qquad}llcl}
① & x + 0 & \to & x & ② & x - 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}
$$

- SN ?

Example

TRS R

$$
\begin{array}{llll}
① & x + 0 \to x & ② & x - 0 \to x \\
③ & x + \mathsf{s}(y) \to \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) \to \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) \to x & ⑥ & \mathsf{s}(\mathsf{p}(x)) \to x
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$

Example

TRS R

$$\begin{array}{llcl}
① & x + 0 & \to & x \\
③ & x + s(y) & \to & s(x + y) \\
⑤ & p(s(x)) & \to & x
\end{array}
\qquad
\begin{array}{llcl}
② & x - 0 & \to & x \\
④ & x - s(y) & \to & p(x - y) \\
⑥ & s(p(x)) & \to & x
\end{array}$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ?

Example

TRS R

$$
\begin{array}{llcl llcl}
\textcircled{1} & x + 0 & \to & x & \textcircled{2} & x - 0 & \to & x \\
\textcircled{3} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \textcircled{4} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\textcircled{5} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \textcircled{6} & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$
\begin{tikzcd}[column sep=small]
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{3}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y))
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{4}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{5}"] & \\
\mathsf{p}(x) & & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{6}"] & \\
\mathsf{s}(x) & & \mathsf{s}(x)
\end{tikzcd}
$$

Example

TRS R

$$
\begin{array}{lrcl@{\qquad}lrcl}
\text{①} & x + 0 & \to & x & \text{②} & x - 0 & \to & x \\
\text{③} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \text{④} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\text{⑤} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \text{⑥} & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$
\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{③}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y))
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{④}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x) & = & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x) & = & \mathsf{s}(x)
\end{tikzcd}
$$

Example

TRS R

$$\begin{array}{lrcllrcl}
\textcircled{1} & x + 0 & \to & x & \textcircled{2} & x - 0 & \to & x \\
\textcircled{3} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \textcircled{4} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\textcircled{5} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \textcircled{6} & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
\textcircled{7} & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & & & &
\end{array}$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{3}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y)) \arrow[ll, "\textcircled{7}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{4}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\textcircled{6}"'] \arrow[dr, "\textcircled{5}"] & \\
\mathsf{p}(x) & = & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{6}"] & \\
\mathsf{s}(x) & = & \mathsf{s}(x)
\end{tikzcd}$$

Example

TRS R

①	$x + 0$	$\to$	x	②	$x - 0$	$\to$	x
③	$x + \mathsf{s}(y)$	$\to$	$\mathsf{s}(x + y)$	④	$x - \mathsf{s}(y)$	$\to$	$\mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x	⑥	$\mathsf{s}(\mathsf{p}(x))$	$\to$	x
⑦	$\mathsf{s}(x + \mathsf{p}(y))$	$\to$	$x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y))$	$\to$	$x - y$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd} & \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "③"] & \\ x + y & & \mathsf{s}(x + \mathsf{p}(y)) \arrow[ll, "⑦"] \end{tikzcd} \qquad \begin{tikzcd} & \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "④"] & \\ x - y & & \mathsf{p}(x - \mathsf{p}(y)) \arrow[ll, "⑧"] \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "⑥"'] \arrow[dr, "⑤"] & \\ \mathsf{p}(x) & = & \mathsf{p}(x) \end{tikzcd} \qquad \begin{tikzcd} & \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "⑤"'] \arrow[dr, "⑥"] & \\ \mathsf{s}(x) & = & \mathsf{s}(x) \end{tikzcd}$$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

- added rewrite rules ⑦ and ⑧ preserve termination

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\leftrightarrow}$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\leftrightarrow}$
- new critical pairs

$$\begin{tikzcd} & \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "⑦"'] \arrow[dr, "⑤"] & \\ \mathsf{p}(x + y) & & x + \mathsf{p}(y) \end{tikzcd} \qquad \begin{tikzcd} & \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑦"] & \\ \mathsf{s}(x + y) & & x + \mathsf{s}(y) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "⑧"'] \arrow[dr, "⑥"] & \\ \mathsf{s}(x - y) & & x - \mathsf{p}(y) \end{tikzcd} \qquad \begin{tikzcd} & \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑧"] & \\ \mathsf{p}(x - y) & & x - \mathsf{s}(y) \end{tikzcd}$$

Example (cont'd)

$$
\begin{array}{clcl}
\textcircled{1} & x + 0 \to x & \textcircled{2} & x + \mathsf{s}(y) \to \mathsf{s}(x + y) \\
\textcircled{3} & x - 0 \to x & \textcircled{4} & x - \mathsf{s}(y) \to \mathsf{p}(x - y) \\
\textcircled{5} & \mathsf{p}(\mathsf{s}(x)) \to x & \textcircled{6} & \mathsf{s}(\mathsf{p}(x)) \to x \\
\textcircled{7} & \mathsf{s}(x + \mathsf{p}(y)) \to x + y & \textcircled{8} & \mathsf{p}(x - \mathsf{p}(y)) \to x - y
\end{array}
$$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\leftrightarrow}$
- new critical pairs

$$
\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "\textcircled{7}"'] \arrow[dr, "\textcircled{5}"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{7}"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "\textcircled{2}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "\textcircled{8}"'] \arrow[dr, "\textcircled{6}"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{8}"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "\textcircled{4}"]
\end{tikzcd}
$$

Example (cont'd)

$$
\begin{array}{rrcl@{\qquad}rrcl}
\textcircled{1} & x + 0 & \to & x & \textcircled{2} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\
\textcircled{3} & x - 0 & \to & x & \textcircled{4} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\textcircled{5} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \textcircled{6} & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
\textcircled{7} & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & \textcircled{8} & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y \\
\textcircled{9} & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y) & & & &
\end{array}
$$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\leftrightarrow}$
- new critical pairs

$$
\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "\textcircled{7}"'] \arrow[dr, "\textcircled{5}"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y) \arrow[ll, "\textcircled{9}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{7}"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "\textcircled{2}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "\textcircled{8}"'] \arrow[dr, "\textcircled{6}"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{8}"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "\textcircled{4}"]
\end{tikzcd}
$$

Example (cont'd)

$$
\begin{array}{clccl}
① & x + 0 \to x & \qquad & ② & x + \mathsf{s}(y) \to \mathsf{s}(x + y) \\
③ & x - 0 \to x & & ④ & x - \mathsf{s}(y) \to \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) \to x & & ⑥ & \mathsf{s}(\mathsf{p}(x)) \to x \\
⑦ & \mathsf{s}(x + \mathsf{p}(y)) \to x + y & & ⑧ & \mathsf{p}(x - \mathsf{p}(y)) \to x - y \\
⑨ & x + \mathsf{p}(y) \to \mathsf{p}(x + y) & & ⑩ & x - \mathsf{p}(y) \to \mathsf{s}(x - y)
\end{array}
$$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\leftrightarrow}$
- new critical pairs

$$
\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "⑦"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y) \arrow[ll, red, "⑨"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑦"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "②"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "⑧"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y) \arrow[ll, red, "⑩"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑧"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "④"]
\end{tikzcd}
$$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- added rewrite rules

  ⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- added rewrite rules

  ⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- rules ⑦ and ⑧ are now superfluous:
  - $\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{⑨} \mathsf{s}(\mathsf{p}(x + y)) \xrightarrow{⑥} x + y$
  - $\mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{⑩} \mathsf{p}(\mathsf{s}(x - y)) \xrightarrow{⑤} x - y$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- added rewrite rules

  ⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- rules ⑦ and ⑧ are now superfluous:
  - $\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{⑨} \mathsf{s}(\mathsf{p}(x + y)) \xrightarrow{⑥} x + y$
  - $\mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{⑩} \mathsf{p}(\mathsf{s}(x - y)) \xrightarrow{⑤} x - y$

  Thus we can drop the rules ⑦ and ⑧.

Example (cont'd)

$$
\begin{array}{rrclrrcl}
① & x + 0 & \to & x & ② & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\
③ & x - 0 & \to & x & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑨ & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y) & ⑩ & x - \mathsf{p}(y) & \to & \mathsf{s}(x - y)
\end{array}
$$

- new critical pairs

$$
\begin{tikzcd}
 & \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "⑤"'] \arrow[dr, "⑨"] & \\
x + y & & \mathsf{p}(x + \mathsf{s}(y))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "⑤"'] \arrow[dr, "⑩"] & \\
x - y & & \mathsf{s}(x - \mathsf{s}(y))
\end{tikzcd}
$$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- new critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{9}"] & \\
x + y & & \mathsf{p}(x + \mathsf{s}(y)) \arrow[dl, "\textcircled{2}"] \\
& \mathsf{p}(\mathsf{s}(x + y)) \arrow[ul, "\textcircled{5}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{10}"] & \\
x - y & & \mathsf{s}(x - \mathsf{s}(y)) \arrow[dl, "\textcircled{4}"] \\
& \mathsf{s}(\mathsf{p}(x - y)) \arrow[ul, "\textcircled{6}"] &
\end{tikzcd}$$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

$$
\begin{array}{llcl@{\qquad}llcl}
① & x + 0 & \to & x & ② & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\
③ & x - 0 & \to & x & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}
$$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑨, ⑩\}$

$$
\begin{array}{lrcl@{\qquad}lrcl}
① & x + 0 & \to & x & ② & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\
③ & x - 0 & \to & x & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑨ & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y) & ⑩ & x - \mathsf{p}(y) & \to & \mathsf{s}(x - y)
\end{array}
$$

- $\mathcal{S}$ is SN LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR all critical pairs of $\mathcal{S}$ are convergent

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\xleftrightarrow[\mathcal{S}]{*} = \xleftrightarrow[R]{*}$

Outline

Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS R such that $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

$R := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ do
- choose $s \approx t \in C \quad C := C \setminus \{s \approx t\}$
- compute R-normal forms s' and t' of s and t
- if $s' \neq t'$ then
  - if $s' > t'$ then
    - $\alpha := s' \quad \beta := t'$
  - else if $t' > s'$ then
    - $\alpha := t' \quad \beta := s'$
  - else
    - *failure* (try another reduction order $>$)
  - $R := R \cup \{\alpha \to \beta\}$
  - $C := C \cup \{e \in \mathrm{CP}(R) \mid \alpha \to \beta \text{ was used to generate } e\}$

Invariants

1. $R \subseteq >$

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2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R \cup C}{\longleftrightarrow}}$

Invariants

1. $R \subseteq >$
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R \cup C]{*}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Invariants

1. $R \subseteq >$
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R \cup C]{*}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure

Invariants

1. $R \subseteq >$
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R \cup C]{*}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*}$

Invariants

1. $R \subseteq >$
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R \cup C}{\longleftrightarrow}}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$
2. terminate with failure

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1. terminate without failure $\implies$ R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$
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3. not terminate (divergence)

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Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad\qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow \rtimes \to \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow \rtimes \to \mathsf{g}(x)$$

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad\qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow \rtimes \rightarrow \mathsf{h}(y) \qquad\qquad \mathsf{h}(y) \leftarrow \rtimes \rightarrow \mathsf{g}(x)$$

- no orientation possible $\Longrightarrow$ failure

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Example

- rewrite rules

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

Three Possibilities

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$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$

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Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad\qquad g(h(a)) \to f(b)$$
$$g(a) \to b$$

- LPO with precedence $a > f > g > h > b$
- critical pair

$$f(b) \leftarrow\rtimes\to g(h(a))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\!\!\!\ltimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow\rtimes\!\!\!\ltimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(h(a)) \to f(b)$$
$$g(a) \to b \qquad g(h(h(a))) \to f(f(b))$$

- LPO with precedence $a > f > g > h > b$
- critical pairs

$$f(b) \leftarrow \rtimes \rightarrow g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \rightarrow g(h(h(a)))$$

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
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$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(a) \to b$$

$$g(h(a)) \to f(b)$$
$$g(h(h(a))) \to f(f(b))$$
$$g(h(h(h(a)))) \to f(f(f(b)))$$
$$\cdots$$

- LPO with precedence $a > f > g > h > b$
- critical pairs

$$f(b) \leftarrow \rtimes \to g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \to g(h(h(a)))$$
$$f(f(f(b))) \leftarrow \rtimes \to g(h(h(h(a))))$$

Outline

- Overview
- Equational Reasoning
- Semantics
- Validity Problem
- Completion
- Efficient Completion

Efficient Completion

Removal of Redundant Rules

In every step of the we are allowed to remove redundant rules from R.

That is, rules $\ell \to r \in R$ such that:

$$\ell \to^*_{R'} r$$

where $R' = R \setminus \{\ell \to r\}$.

Efficient Completion

Removal of Redundant Rules

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That is, rules $\ell \to r \in R$ such that:

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Observation

- fewer rewrite rules $\Longrightarrow$ fewer critical pairs