ISR 2026 — Basic Track

Jörg Endrullis
Cynthia Kop
Framke van Raamsdonk

partially based on slides by: Aart Middeldorp

hosting institution:
Radboud Universiteit Nijmegen
The Netherlands

- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

- Overview
- Motivation
- Dependency pairs and chains: definition
- Using a reduction pair
- A modular framework
- Other processors

Recall: subterm property

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\mathsf{minus}(x, 0) & \to & x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
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If $\mathsf{minus}(x, y) \succ y$ and $\mathsf{s}(a) \succ a$, then:

$$\mathsf{s}(\mathsf{quot}(\underline{\mathsf{minus}(x, y)}, \mathsf{s}(y))) \quad \succ \quad \mathsf{s}(\mathsf{quot}(\underline{y}, \mathsf{s}(y)))$$

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Definition (Subterm Property)

A reduction ordering $\succ$ has the subterm property if $\mathsf{f}(\dots, s, \dots) \succeq s$ for all f.

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Wish: split a termination problem into multiple smaller problems.

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The core idea of dependency pairs is to look at function calls.

To start, split functions in constructors and defined symbols.

Identifying function calls

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\end{aligned}$$

We isolate the calls from one defined symbol to another:

$$\begin{aligned}
\mathsf{minus}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\Rightarrow \mathsf{minus}^\sharp(x, y) \\
\mathsf{quot}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\Rightarrow \mathsf{quot}^\sharp(\mathsf{minus}(x, y), \mathsf{s}(y)) \\
\mathsf{quot}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\Rightarrow \mathsf{minus}^\sharp(x, y)
\end{aligned}$$

Dependency pair chains

We can prove the following result:

Theorem (Dependency Pair Chains)

For a given set of rules R, let DP be the corresponding set of dependency pairs.

$\to_R$ is terminating if and only if there is no infinite (DP, R)-chain: a reduction $s_1 \Rightarrow_{DP} \to_R^* s_2 \Rightarrow_{DP} \to_R^* s_3 \ldots$

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So $\to_R$ is terminating iff there is no infinite sequence where:

- the steps using $\Rightarrow_{\mathrm{DP}}$ occur at the root of the term;
- the steps using $\to_R$ do not occur at the root of the term;
- there are infinitely many steps using $\Rightarrow_{\mathrm{DP}}$.

Example: an infinite DP chain

$$R = \left\{ \begin{array}{rcl} \mathsf{f}(0, x) & \to & \mathsf{g}(\mathsf{f}(\mathsf{g}(x), x)) \\ \mathsf{g}(x) & \to & x \end{array} \right\}$$

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Infinite chain: $\underline{\mathsf{f}^\sharp(0,0)} \Rightarrow_{\mathrm{DP}} \mathsf{f}^\sharp(\underline{\mathsf{g}(0)},0) \to_R \underline{\mathsf{f}^\sharp(0,0)} \Rightarrow_{\mathrm{DP}} \mathsf{f}^\sharp(\underline{\mathsf{g}(0)},0) \to_R \ldots$

Proving the dependency pair theorem

Theorem: $\to_R$ is terminating if and only if there is no infinite reduction $s_1 \Rightarrow_{\mathsf{DP}} \to_R^* s_2 \Rightarrow_{\mathsf{DP}} \to_R^* s_3 \ldots$

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- Note that $s \Rightarrow_{\mathsf{DP}} t$ implies $s \to_R s' \trianglerighteq t$ for some s'.
- Hence, there is an infinite chain
  $s_1 \to_R C_1[t_1] \to_R^* C_1[s_2] \to_R C_1[C_2[t_2]] \to_R^* C_1[C_2[s_3]] \dots$

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- Then necessarily: $\mathsf{g}(t_1, \ldots, t_m) = r'\sigma$ for some $r' \trianglelefteq r$

Building block: reduction pairs

How do you prove that there is no infinite chain?

then there is no infinite (DP, R)-chain!

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One possibility: using a variant of a reduction ordering: a reduction pair.

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Proof idea. By these requirements:

- If $s \Rightarrow_{\mathsf{DP}} t$ (by a step at the root), then $s \succ t$.
- If $s \to_R t$ then $s \succeq t$.
- Hence, if $s \Rightarrow_{\mathsf{DP}} \cdot \to_R^* t$, then $s \succ \cdot \succeq^* t$, and therefore $s \succ t$.

□

Quot/minus: ordering requirements

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$$\text{if } s \geq t, \text{ then } [\mathsf{f}](\ldots, s, \ldots) \geq [\mathsf{f}](\ldots, t, \ldots).$$

Many functions that are not monotonic, are still weakly monotonic; for example:

- functions that ignore arguments: $\lambda x, y.\ x$
- min / max functions: $\lambda x, y.\ \min(x, y)$ or $\lambda x.\ \max(x - 1, 0)$

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Back to our example!

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\end{array}$$

This is satisfied by choosing:

$$\begin{array}{rclrclrcl}
[0] & = & 0 & [\mathsf{minus}] & = & \lambda x, y.\ x & [\mathsf{minus}^\sharp] & = & \lambda x, y.\ x \\
[\mathsf{s}] & = & \lambda x.\ x + 1 & [\mathsf{quot}] & = & \lambda x, y.\ x & [\mathsf{quot}^\sharp] & = & \lambda x, y.\ x
\end{array}$$

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Back to our example!

$$\begin{array}{rcl} x & \geq & x \\ x+1 & \geq & x \\ 0 & \geq & 0 \\ x+1 & \geq & x+1 \\ x+1 & > & x \\ x+1 & > & x \\ x+1 & > & x \end{array}$$

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Hence, we clearly gained something!

Step-by-step proofs

Challenge: dealing with large systems.

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Example:

Rules:

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\end{aligned}$$

DPs:

$$\begin{aligned}
\mathsf{A}\quad \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\Rightarrow \mathsf{minus}^\sharp(x,y)\\
\mathsf{B}\quad \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\Rightarrow \mathsf{minus}^\sharp(x,y)\\
\mathsf{C}\quad \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\Rightarrow \mathsf{quot}^\sharp(\mathsf{minus}(x,y),\mathsf{s}(y))
\end{aligned}$$

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Rules:		$\mathsf{minus}(x, 0)$	$\to$	x
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		$\mathsf{quot}(0, \mathsf{s}(y))$	$\to$	0
		$\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y))$	$\to$	$\mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y)))$
DPs:	A	$\mathsf{minus}^\sharp(\mathsf{s}(x), \mathsf{s}(y))$	$\Rightarrow$	$\mathsf{minus}^\sharp(x, y)$
	B	$\mathsf{quot}^\sharp(\mathsf{s}(x), \mathsf{s}(y))$	$\Rightarrow$	$\mathsf{minus}^\sharp(x, y)$
	C	$\mathsf{quot}^\sharp(\mathsf{s}(x), \mathsf{s}(y))$	$\Rightarrow$	$\mathsf{quot}^\sharp(\mathsf{minus}(x, y), \mathsf{s}(y))$

Question: what does an infinite chain look like?

Step-by-step proofs

Challenge: dealing with large systems.

Idea: split large problems into smaller sub-problems.

Example:

Rules:		$\mathsf{minus}(x, 0)$	$\to$	x
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Question: what does an infinite chain look like?

We conclude: split DP into $\{A\}$ and $\{C\}$!

Using a graph

Using a graph

Idea: Graph with DPs as nodes, edges between nodes if one can follow another in a chain.

Using a graph

Idea: Graph with DPs as nodes, edges between nodes if one can follow another in a chain.

Example:

$$\begin{array}{lrcl} \text{A.} & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \Rightarrow & \mathsf{minus}^\sharp(x,y) \\ \text{B.} & \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \Rightarrow & \mathsf{minus}^\sharp(x,y) \\ \text{C.} & \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \Rightarrow & \mathsf{quot}^\sharp(\mathsf{minus}(x,y),\mathsf{s}(y)) \end{array}$$

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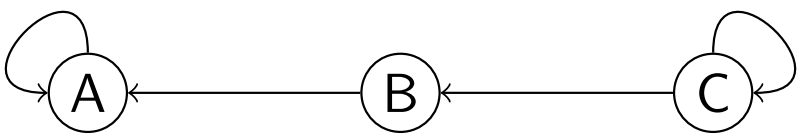
\end{array}
$$



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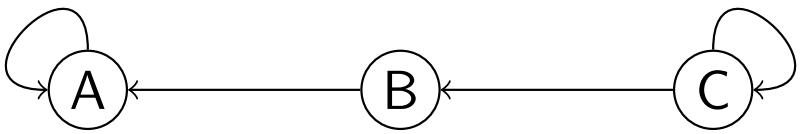

Observation: each strongly connected component may be considered separately.

Dependency graph processor – another example

$$R = \{\mathsf{f}(\mathsf{f}(x)) \ \rightarrow \ \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))\}$$

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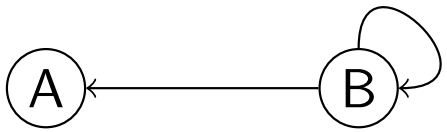

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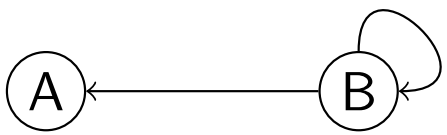

Hence, we can remove dependency pair A.

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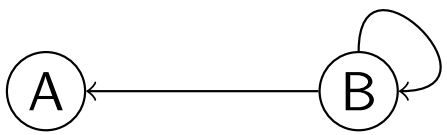

Hence, we can remove dependency pair A.

(And then quickly remove B, too, using weakly monotonic algebras.)

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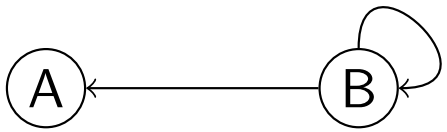

Hence, we can remove dependency pair A.

(And then quickly remove B, too, using weakly monotonic algebras.)

Note: not always easy to see if one DP can follow another, but we can use approximations.

The DP framework visualised

We may visualise an example termination proof by the DP framework as follows:

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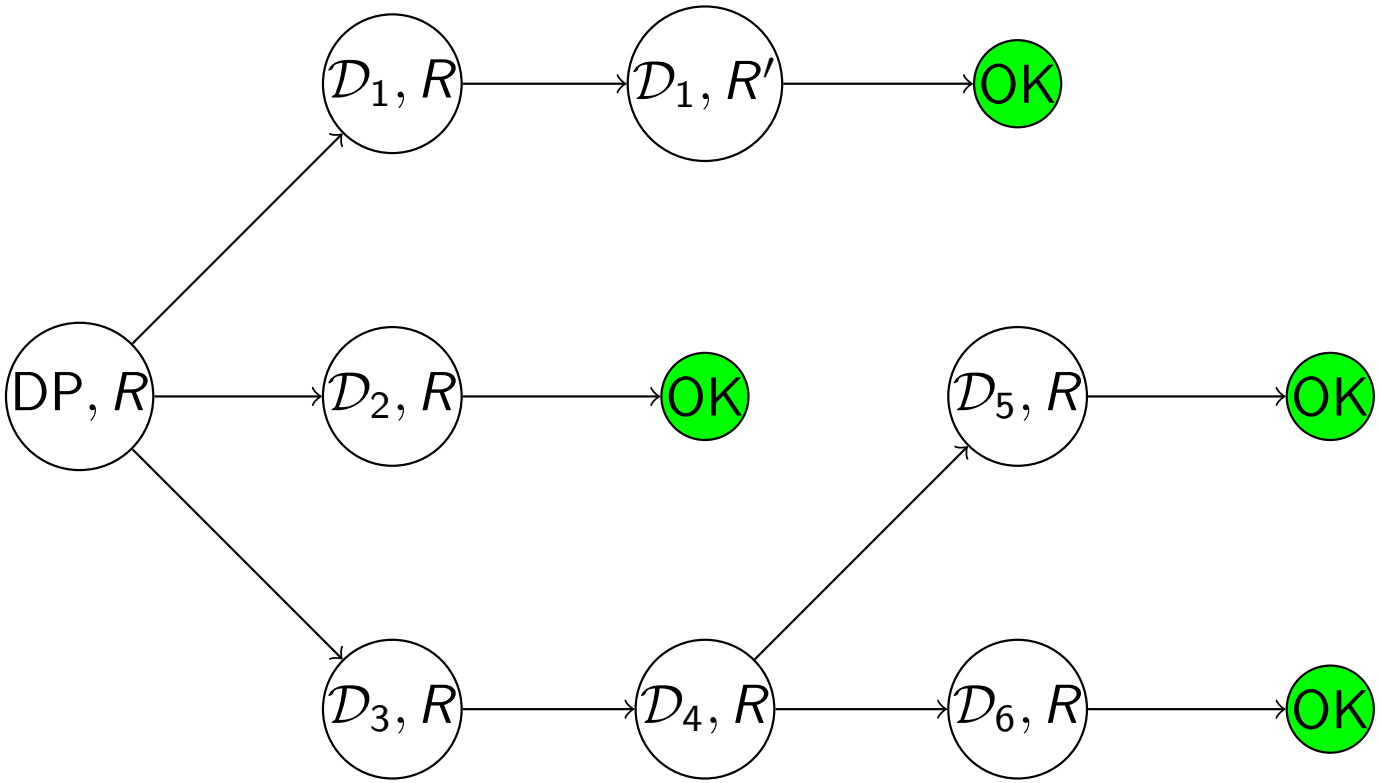

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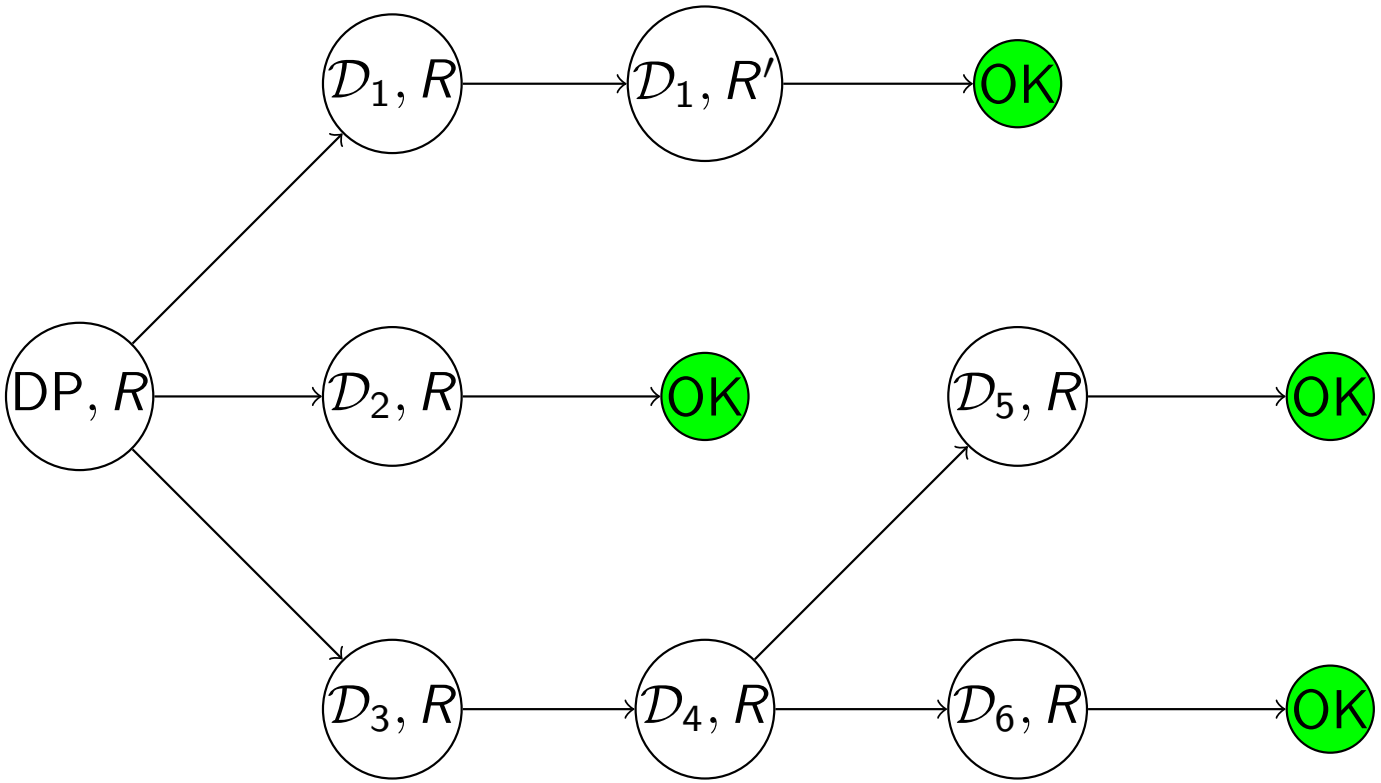

A pair $(\mathcal{D}, R)$ is called a DP problem, and a method to alter a DP problem is a DP processor.

The subterm criterion

$$
\begin{array}{lrcl}
\text{A.} & \exp^\sharp(\mathsf{s}(x), y) & \Rightarrow & \mathrm{double}^\sharp(x, y, 0) \\
\text{B.} & \mathrm{double}^\sharp(x, 0, z) & \Rightarrow & \exp^\sharp(x, z) \\
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\end{array}
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Idea: consider the first argument of each side of the dependency pairs.

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Idea: consider the first argument of each side of the dependency pairs.

Then we can remove the dependency pairs where the chosen argument becomes smaller (in this case A).

The subterm criterion processor

Definition

For all marked symbols $\mathsf{f}^\sharp$, let $\nu(\mathsf{f}^\sharp) \in \{1, \ldots, \mathit{arity}(\mathsf{f})\}$.

Let $(\mathcal{D}, R)$ be a DP problem, and write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Suppose:

- $\ell_{\nu(\mathsf{f}^\sharp)} = r_{\nu(\mathsf{g}^\sharp)}$ for all $\mathsf{f}^\sharp(\ell_1, \ldots, \ell_n) \Rightarrow \mathsf{g}^\sharp(r_1, \ldots, r_m) \in \mathcal{D}_1$
- $r_{\nu(\mathsf{f}^\sharp)}$ is a subterm of $\ell_{\nu(\mathsf{g}^\sharp)}$ for all $\mathsf{f}^\sharp(\ell_1, \ldots, \ell_n) \Rightarrow \mathsf{g}^\sharp(r_1, \ldots, r_m) \in \mathcal{D}_2$

Then the subterm criterion processor maps $(\mathcal{D}, R)$ to $\{(\mathcal{D}_1, R)\}$.

Graph + subterm criterion

Class exercise:

$$
\begin{array}{rcl}
\mathsf{exp}(0, y) & \to & y \\
\mathsf{exp}(\mathsf{s}(x), y) & \to & \mathsf{double}(0, x, y) \\
\mathsf{double}(r, x, 0) & \to & \mathsf{exp}(x, r) \\
\mathsf{double}(r, x, \mathsf{s}(y)) & \to & \mathsf{double}(\mathsf{s}(\mathsf{s}(r)), x, y)
\end{array}
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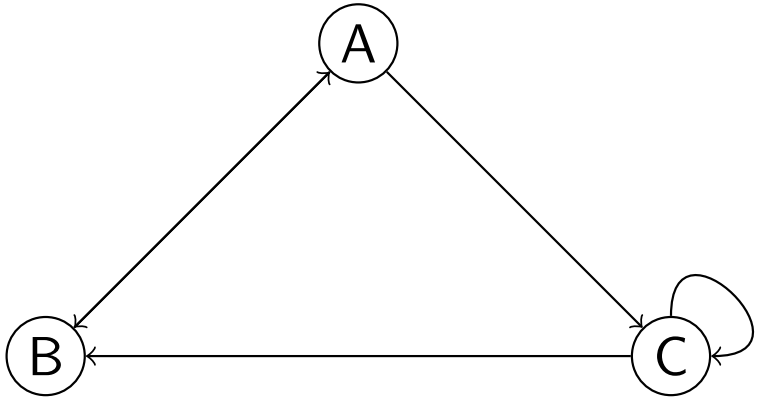

Graph + subterm criterion

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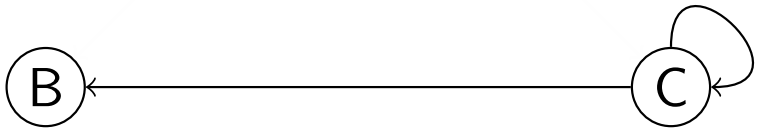

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Reduction pair processor

We can also reformulate reduction pairs as a processor:

Definition (Reduction Pair Processor)

Let $\succ$ be a well-founded, stable ordering and $\succeq$ a stable monotonic quasi-ordering on terms, such that $\succ\succeq \subseteq \succ$.

Let $(\mathcal{D}, R)$ be a DP problem, and write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Suppose:

- $\ell \succeq r$ for all $\ell \to r \in R$
- $\ell \succeq r$ for all $\ell \Rightarrow r \in \mathcal{D}_1$, and
- $\ell \succ r$ for all $\ell \Rightarrow r \in \mathcal{D}_2$.

Then the reduction pair processor maps $(\mathcal{D}, R)$ to $\{(\mathcal{D}_1, R)\}$.

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- $\ell \succ r$ for all $\ell \Rightarrow r \in \mathcal{D}_2$.

Then the reduction pair processor maps $(\mathcal{D}, R)$ to $\{(\mathcal{D}_1, R)\}$.

That is, we can use a reduction pair, and remove all dependency pairs that were ordered with $\succ$.

Using a reduction pair processor

$$
\begin{array}{rcl}
\mathsf{append}(\mathsf{nil}, z) & \to & z \\
\mathsf{append}(\mathsf{cons}(x, y), z) & \to & \mathsf{cons}(x, \mathsf{append}(y, z)) \\
\mathsf{rev}(\mathsf{nil}) & \to & \mathsf{nil} \\
\mathsf{rev}(\mathsf{cons}(x, y)) & \to & \mathsf{append}(\mathsf{rev}(y), \mathsf{cons}(x, \mathsf{nil})) \\
\mathsf{append}^\sharp(\mathsf{cons}(x, y), z) & \Rightarrow & \mathsf{append}^\sharp(y, z) \\
\mathsf{rev}^\sharp(\mathsf{cons}(x, y)) & \Rightarrow & \mathsf{rev}^\sharp(y) \\
\mathsf{rev}^\sharp(\mathsf{cons}(x, y)) & \Rightarrow & \mathsf{append}^\sharp(\mathsf{rev}(y), \mathsf{cons}(x, \mathsf{nil}))
\end{array}
$$

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\mathsf{rev}(\mathsf{cons}(x, y)) & \to & \mathsf{append}(\mathsf{rev}(y), \mathsf{cons}(x, \mathsf{nil})) \\
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\mathsf{rev}^\sharp(\mathsf{cons}(x, y)) & \Rightarrow & \mathsf{rev}^\sharp(y) \\
\mathsf{rev}^\sharp(\mathsf{cons}(x, y)) & \Rightarrow & \mathsf{append}^\sharp(\mathsf{rev}(y), \mathsf{cons}(x, \mathsf{nil}))
\end{array}
$$

We choose:

$$
\begin{array}{rcl}
[\mathsf{nil}] & = & 0 \\
[\mathsf{cons}] & = & \lambda x, y.\ y + 1
\end{array}
\qquad
\begin{array}{rcll}
[\mathsf{append}] & = & \lambda x, y. & x + y \\
[\mathsf{rev}] & = & \lambda x. & x \\
[\mathsf{append}^\sharp] & = & \lambda x, y. & x + y \\
[\mathsf{rev}^\sharp] & = & \lambda x. & x
\end{array}
$$

Using a reduction pair processor

$$\begin{array}{rcl} z & \geq & z \\ (y+1)+z & \geq & (y+z)+1 \\ 0 & \geq & 0 \\ y+1 & \geq & y+(0+1) \\ (y+1)+z & > & y+z \\ y+1 & > & y \\ y+1 & \geq & y+(0+1) \end{array}$$

We choose:

$$\begin{array}{rcl} [\mathsf{nil}] & = & 0 \\ [\mathsf{cons}] & = & \lambda x, y.\ y+1 \end{array} \qquad \begin{array}{rcl} [\mathsf{append}] & = & \lambda x, y.\ x+y \\ [\mathsf{rev}] & = & \lambda x.\ x \\ [\mathsf{append}^\sharp] & = & \lambda x, y.\ x+y \\ [\mathsf{rev}^\sharp] & = & \lambda x.\ x \end{array}$$

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Hence, we can remove all but the last dependency pair, and continue with:

$$(\{\mathsf{rev}^\sharp(\mathsf{cons}(x,y)) \Rightarrow \mathsf{append}^\sharp(\mathsf{rev}(y),\mathsf{cons}(x,\mathsf{nil}))\},R)$$

Argument filterings

Idea: remove arguments before using a reduction ordering

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\begin{array}{rcl}
\mathsf{minus}(x, 0) & \succeq & x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \succeq & \mathsf{minus}(x, y) \\
\mathsf{quot}(0, \mathsf{s}(y)) & \succeq & 0 \\
\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \succeq & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \\
\mathsf{minus}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) & \succ & \mathsf{minus}^\sharp(x, y) \\
\mathsf{quot}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) & \succ & \mathsf{quot}^\sharp(\mathsf{minus}(x, y), \mathsf{s}(y)) \\
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\mathsf{quot}(\mathsf{s}(x),\mathsf{s}(y)) & \succeq & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
\mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \succ & \mathsf{minus}^\sharp(x,y) \\
\mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \succ & \mathsf{quot}^\sharp(\mathsf{minus}(x,y),\mathsf{s}(y)) \\
\mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \succ & \mathsf{minus}^\sharp(x,y)
\end{array}
$$

Replace $\mathsf{minus}(x,y)$ by $\mathsf{minus}'(x)$.

Argument filterings

Idea: remove arguments before using a reduction ordering

$$
\begin{array}{rcl}
\mathsf{minus}'(x) & \trianglerighteq & x \\
\mathsf{minus}'(\mathsf{s}(x)) & \trianglerighteq & \mathsf{minus}'(x) \\
\mathsf{quot}(0, \mathsf{s}(y)) & \trianglerighteq & 0 \\
\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \trianglerighteq & \mathsf{s}(\mathsf{quot}(\mathsf{minus}'(x), \mathsf{s}(y))) \\
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Idea: remove arguments before using a reduction ordering

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Searching a filter can be included in the SAT encoding of LPO.

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Finding usable rules is a very simple reachability algorithm.

More complex (but powerful!): combining usable rules with an argument filtering and a reduction pair.