ISR 2026 — Basic Track

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partially based on slides by: Aart Middeldorp

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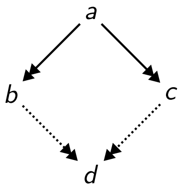
The Netherlands

- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Weak Church–Rosser

Church–Rosser vs Weak Church–Rosser



Church–Rosser (Confluence)

Church–Rosser vs Weak Church–Rosser

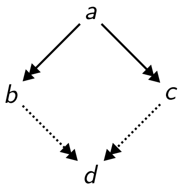
$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\ & d & \end{tikzcd}$$

Church–Rosser (Confluence)

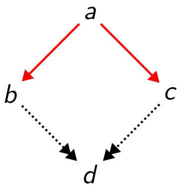
$$\begin{tikzcd} & a \arrow[dl, red] \arrow[dr, red] & \\ b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\ & d & \end{tikzcd}$$

Weak Church–Rosser (Local confluence)

Church–Rosser vs Weak Church–Rosser



Church–Rosser (Confluence)



Weak Church–Rosser (Local confluence)

Can we conclude Church–Rosser from Weak Church–Rosser?

Church–Rosser vs Weak Church–Rosser

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\
 & d &
\end{tikzcd}
$$

Church–Rosser
(Confluence)

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\
 & d &
\end{tikzcd}
$$

Weak Church–Rosser
(Local confluence)

Can we conclude Church–Rosser from Weak Church–Rosser?

Not in general:

$$
\begin{tikzcd}
\mathrm{a} & \mathrm{b} \arrow[l] \arrow[r, bend left] & \mathrm{c} \arrow[l, bend left] \arrow[r] & \mathrm{d}
\end{tikzcd}
$$

We need further conditions! **Idea: label the steps!**

Indexed Abstract Rewrite Systems

ARSs with Multiple Relations

Definitions

- abstract rewrite system (ARS) is set A with binary relations $\rightarrow_i$ for $i \in \mathcal{I}$

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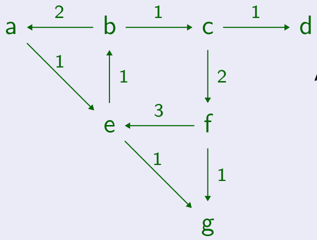
$$\begin{tikzcd}
a \arrow[dr, "1"] & b \arrow[l, "2"'] \arrow[r, "1"] & c \arrow[r, "1"] \arrow[d, "2"] & d \\
 & e \arrow[u, "1"] \arrow[dr, "1"] & f \arrow[l, "3"'] \arrow[d, "1"] & \\
 & & g &
\end{tikzcd}$$

ARSs with Multiple Relations

Definitions

- abstract rewrite system (ARS) is set A with binary relations $\to_i$ for $i \in \mathcal{I}$



ARS $\mathcal{A} = \langle A, \to_1, \to_2, \to_3 \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to_1 = \left\{ \begin{array}{l} (a, e), (b, c), (c, d), \\ (e, b), (e, g), (f, g) \end{array} \right\}$
- $\to_2 = \{(b, a), (c, f)\}$
- $\to_3 = \{(f, e)\}$

- $\to \; = \; \to_1 \cup \to_2 \cup \to_3$

Definition

Let $\mathcal{A} = \langle A, \rightarrow_1, \rightarrow_2 \rangle$ be an ARS.

- $\rightarrow_1$ commutes with $\rightarrow_2$
  - ${}_2^*\!\leftarrow \cdot \rightarrow_1^* \;\subseteq\; \rightarrow_1^* \cdot {}_2^*\!\leftarrow$

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  - $\forall a, b, c$

$$\begin{tikzcd}
& a \arrow[dl, "2"'] \arrow[dr, "1"] & \\
b \arrow[dr, dotted, "1"'] & & c \arrow[dl, dotted, "2"] \\
& d &
\end{tikzcd}$$

$\exists d$

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    - $\forall a, b, c$ ... $\exists d$

$$
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 & d &
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$$

- $\to_1$ commutes weakly with $\to_2$
    - ${}_2\!\leftarrow \cdot \to_1 \;\subseteq\; \to_1^* \cdot {}_2^*\!\leftarrow$

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Decreasing Diagrams for Confluence

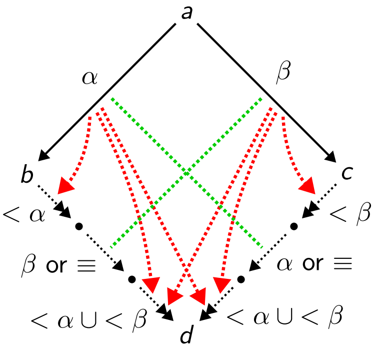
Decreasing Diagrams

We **label the steps** $\to$ with a labels from a **well-founded partial order** $(I, <)$:

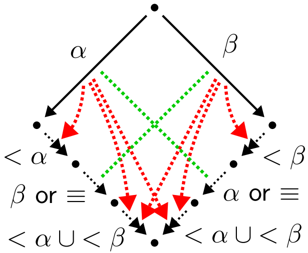
$$\to \; = \bigcup_{\alpha \in I} \to_\alpha \qquad \to_{<\beta} \; = \bigcup_{\alpha < \beta} \to_\alpha \qquad \to_{\leq\beta} \; = \bigcup_{\alpha \leq \beta} \to_\alpha$$

Decreasing Diagrams [van Oostrom]

CR($\to$) if every peak $b \leftarrow_\alpha a \to_\beta c$ admits a **decreasing diagram** (shown below).



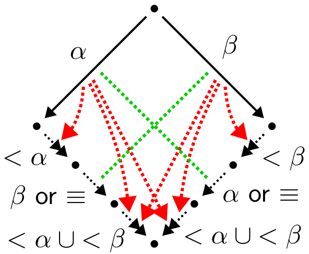
Example: Grid



How to label the following system?

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & c \arrow[d] \\
d \arrow[r] \arrow[d] & e \arrow[r] \arrow[d] & f \arrow[d] \\
g \arrow[r] & h \arrow[r] & i
\end{tikzcd}$$

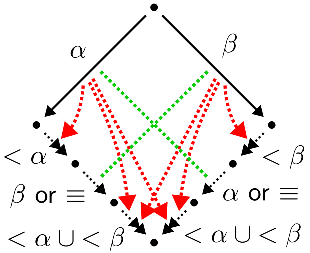
Example: Grid



How to label the following system? We use labels $(\{0\}, <)$ as follows:

$$
\begin{tikzcd}
a \arrow[r, "0"] \arrow[d, "0"'] & b \arrow[r, "0"] \arrow[d, "0"'] & c \arrow[d, "0"'] \\
d \arrow[r, "0"] \arrow[d, "0"'] & e \arrow[r, "0"] \arrow[d, "0"'] & f \arrow[d, "0"'] \\
g \arrow[r, "0"] & h \arrow[r, "0"] & i
\end{tikzcd}
$$

Diamond Property



Diamond property: CR($\to$) if for all peaks $b \leftarrow a \rightarrow c$:

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\ & d & \end{tikzcd}$$

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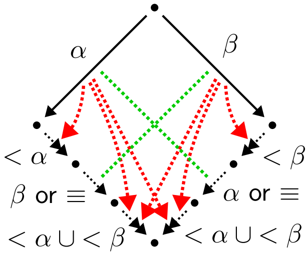
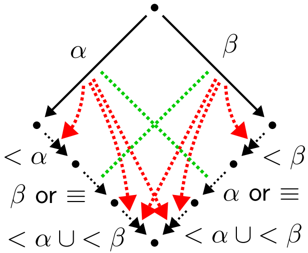


Diamond property: CR($\to$) if for all peaks $b \leftarrow a \to c$:

$$\begin{tikzcd} & a \arrow[dl, "0"'] \arrow[dr, "0"] & \\ b \arrow[dr, dotted, "0"'] & & c \arrow[dl, dotted, "0"] \\ & d & \end{tikzcd}$$

Decreasing diagrams implies the diamond property: labels $(\{ 0 \}, <)$.

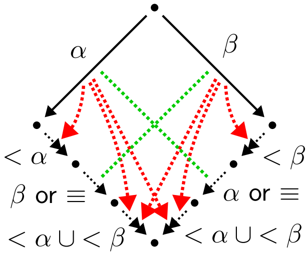
Subcommutativity



Subcommutativity: CR($\rightarrow$) if for all peaks $b \leftarrow a \rightarrow c$:

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted, "="'] & & c \arrow[dl, dotted, "="] \\
 & d &
\end{tikzcd}$$

Subcommutativity



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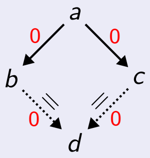
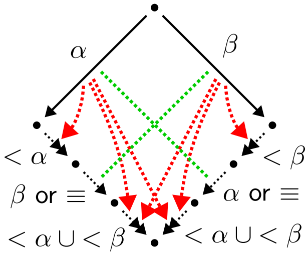


Decreasing diagrams implies the subcommutativity: labels $(\{\, 0 \,\}, <)$.

Example: Infinite Branching



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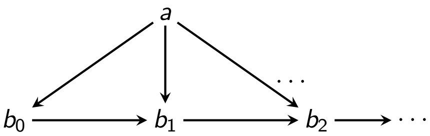

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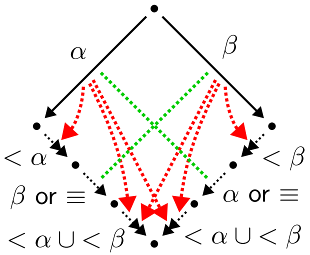
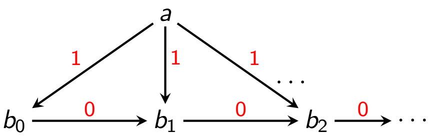
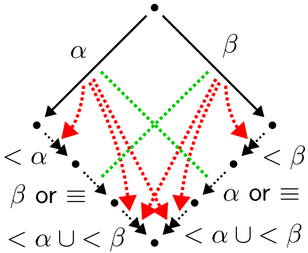


How to label the following system? We use labels $(\{0, 1\}, <)$ as follows:



Example: Nested Peaks



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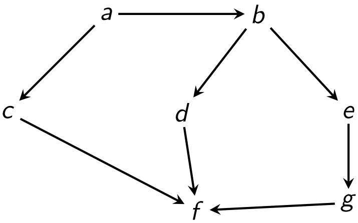

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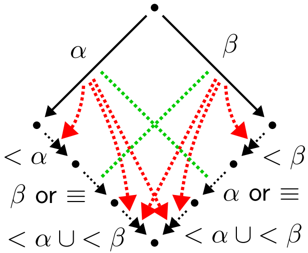

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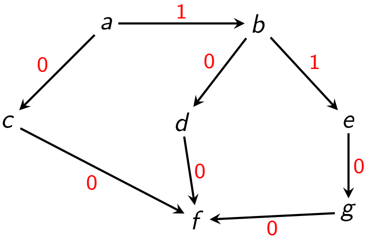
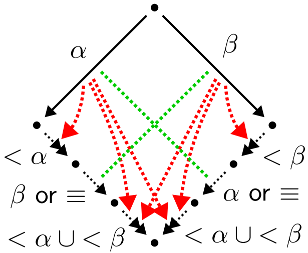


Newman's Lemma



Newmans Lemma (NL): CR($\rightarrow$) if WCR($\rightarrow$) and SN($\rightarrow$).

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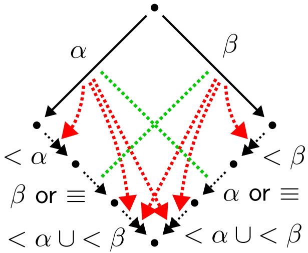
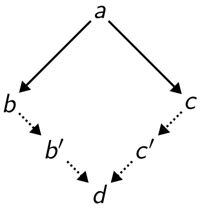


Newmans Lemma (NL): CR($\to$) if WCR($\to$) and SN($\to$).

Example peak:



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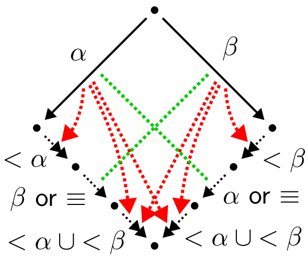

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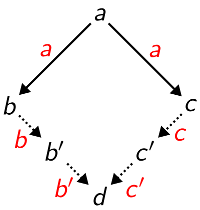
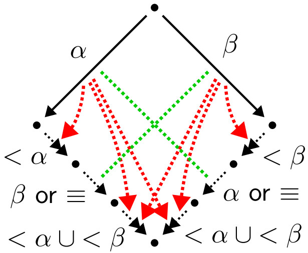


Decreasing diagrams implies NL: labels $(A, \to^+)$ and label $a \to b$ with a.

Rosen's Requests Lemma



Requests Lemma: $\mathrm{CR}(\rightarrow_1 \cup \rightarrow_2)$ if

$$
\begin{tikzcd}
 & a \arrow[dl, "1"'] \arrow[dr, "1"] & \\
b \arrow[dr, dotted, "1"', "\equiv"] & & c \arrow[dl, dotted, "1", "\equiv"'] \\
 & d &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[dl, "2"'] \arrow[dr, "2"] & \\
b \arrow[dr, dotted, "2"', "\equiv"] & & c \arrow[dl, dotted, "2", "\equiv"'] \\
 & d &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[dl, "1"'] \arrow[dr, "2"] & \\
b \arrow[d, dotted, "2"'] & & c \arrow[ddl, dotted, "1"] \\
\bullet \arrow[dr, dotted, "1"'] & & \\
 & d &
\end{tikzcd}
$$

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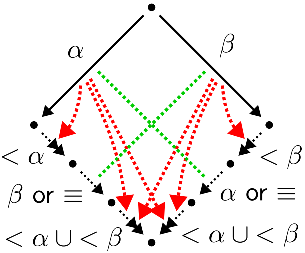

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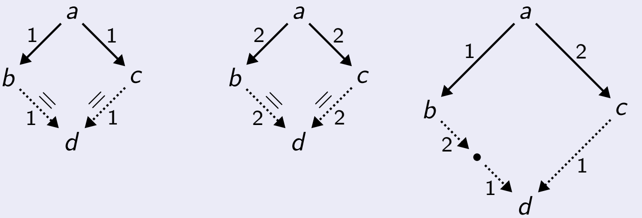

Decreasing diagrams implies the request lemma: labels $(\{1, 2\}, <)$.

Countable Systems: Cofinality Property

A set A is **countable** if A is empty or there is a surjection $c : \mathbb{N} \to A$.

Every countable, connected, confluent $(A, \to)$ has a **main road**: a rewrite sequence

$$m_0 \to m_1 \to m_2 \to \ldots$$

such that every $a \in A$ rewrites to some m_i on the main road.

Let $c : \mathbb{N} \to A$ be surjective.

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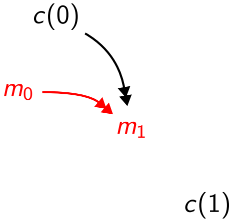

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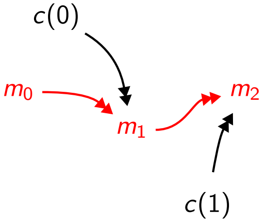

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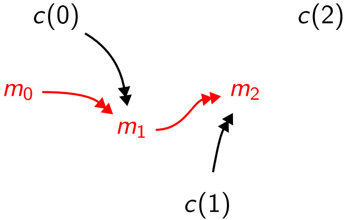

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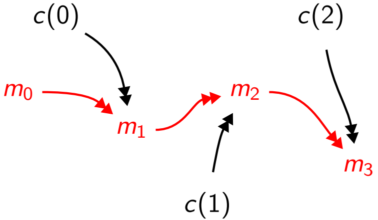
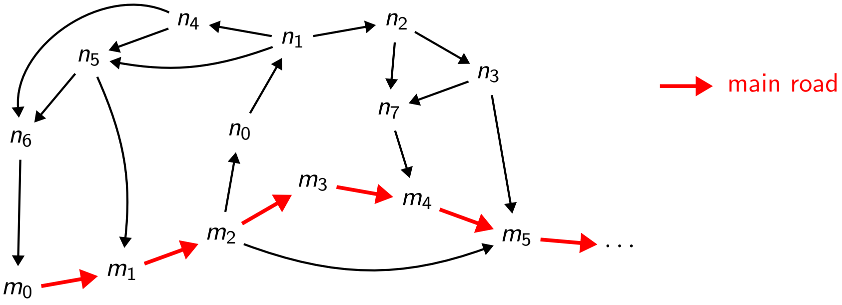


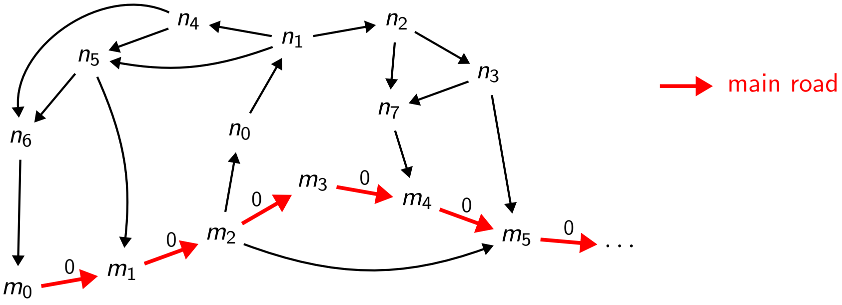
We may assume that the
main road has no cycles.

Countable Systems



How to label a system with a main road?

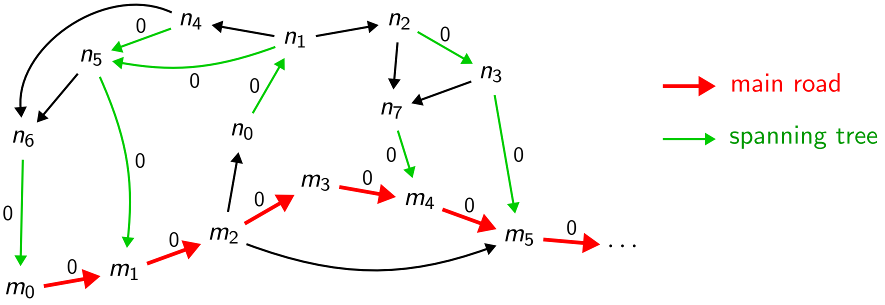
Countable Systems



How to label a system with a main road?

- main road gets label 0

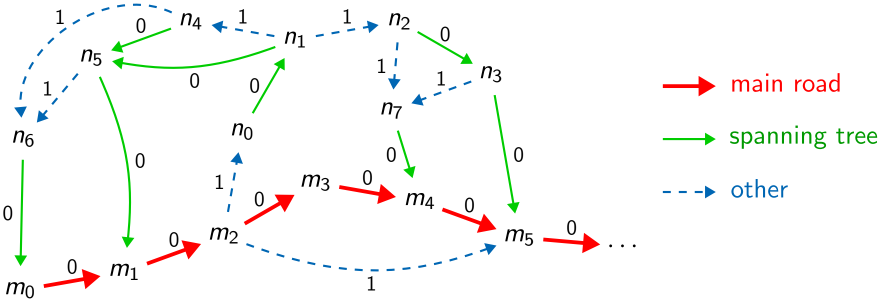
Countable Systems



How to label a system with a main road?

- main road gets label 0
- for every element: one step on a shortest path to the main road gets label 0

Countable Systems



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- all other steps get label 1

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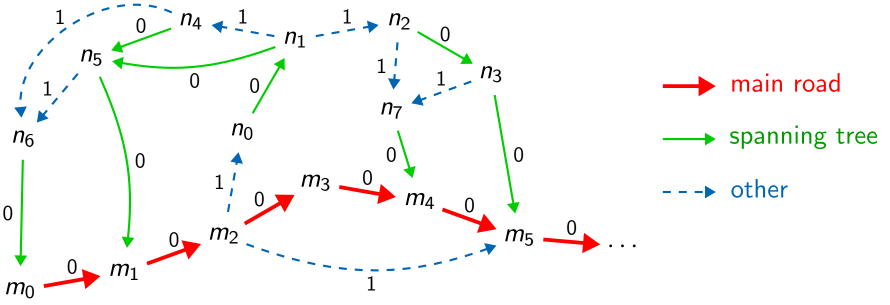

How to label a system with a main road?

- main road gets label 0
- for every element: one step on a shortest path to the main road gets label 0
- all other steps get label 1

All confluent, countable systems admit a labelling with $(\{0, 1\}, <)$.

Example: Uncountable Systems [Ivanov 2024]

Consider the following ARS $(A, \to)$:

$$A = \{\, 1, 2, 3 \,\} \times \mathcal{P}_{fin}(\mathbb{R})$$

$$\begin{aligned} (1, P) &\to (2, P \uplus \{\, p, q \,\}) \\ (2, P) &\to (3, P) \\ (3, P) &\to (2, P \uplus \{\, p \,\}) \end{aligned}$$

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Peaks:

$$\begin{tikzcd} & (3, P) \arrow[dl] \arrow[dr] & \\ (2, P \uplus \{\, p \,\}) & & (2, P \uplus \{\, p' \,\}) \end{tikzcd}$$

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Peaks:

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(3, P \uplus \{\, p \,\}) \arrow[dr, "\text{problem: } 0"'] & & (3, P \uplus \{\, p' \,\}) \arrow[dl, "0"] \\
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Example: Uncountable Systems [Ivanov 2024]

Consider the following ARS $(A, \rightarrow)$:

$$A = \{ 1, 2, 3 \} \times \mathcal{P}_{fin}(\mathbb{R})$$

$$\begin{aligned}
(1, P) &\rightarrow (2, P \uplus \{ p, q \}) \\
(2, P) &\xrightarrow{0} (3, P) \\
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\end{aligned}$$

Peaks:

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\end{tikzcd}$$

Example: Uncountable Systems [Ivanov 2024]

Consider the following ARS $(A, \to)$: **This ARS needs 3 labels**!

$$A = \{\, 1, 2, 3 \,\} \times \mathcal{P}_{fin}(\mathbb{R})$$

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Peaks:

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(3, P \uplus \{\, p, q \,\}) \arrow[d, "1"'] & & (3, P \uplus \{\, p', q' \,\}) \arrow[d, "1"] \\
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\end{tikzcd}$$

Commutation

Confluence via Commutation

$$\begin{tikzcd}[column sep=small] & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr, dotted, ->>] & & c \arrow[dl, dotted, ->>] \\ & d & \end{tikzcd}$$

Church–Rosser (Confluence)

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, red] \arrow[dr, blue] & \\ b \arrow[dr, dotted, ->>, blue] & & c \arrow[dl, dotted, ->>, red] \\ & d & \end{tikzcd}$$

Commutation (2 Relations)

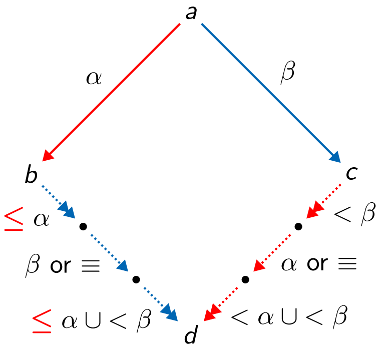
$\mathrm{CR}(\to) \quad\Longleftrightarrow\quad \to$ commutes with $\to$

Decreasing Diagrams for Commutation

Decreasing Diagrams for Commutation

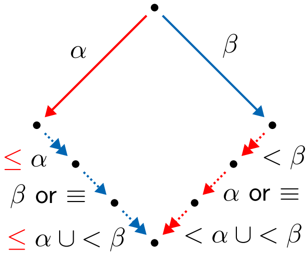
Decreasing Diagrams for Commutation [van Oostrom]

$\color{red}{\rightarrow}$ commutes with $\color{blue}{\rightarrow}$ if every peak $b \mathrel{\color{red}{\leftarrow_\alpha}} a \mathrel{\color{blue}{\rightarrow_\beta}} c$ admits a **decreasing diagram**:

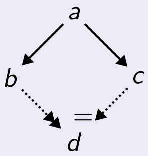


Observe the relaxed conditions $\leq \alpha$ instead of $< \alpha$!

Huet's Strong Confluence



Strong confluence: CR($\to$) if for all peaks $\leftarrow \cdot \to \subseteq \to^* \cdot \leftarrow^=$



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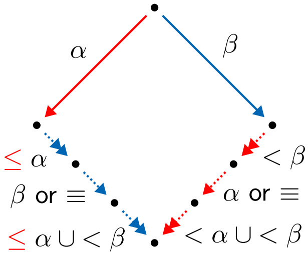
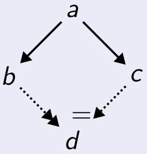
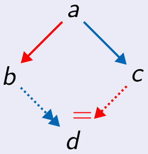


Strong confluence: CR($\rightarrow$) if for all peaks $\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot \leftarrow^=$





Strong commutation: $\rightarrow$ commutes with $\rightarrow$ if $\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot \leftarrow^=$

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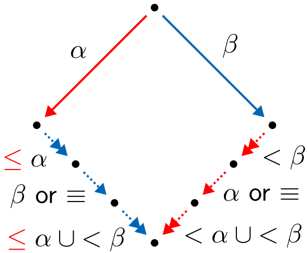

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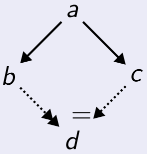

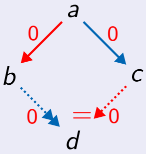

Strong commutation: $\to$ commutes with $\to$ if $\leftarrow \cdot \to \subseteq \to^* \cdot \leftarrow^=$

Decreasing diagrams implies strong commutation: labels $(\{0\}, <)$.

Decreasing Diagrams and Term Rewriting

Rule Labelling [van Oostrom 2008]

Confluence criterion

Let R be a left- and right-linear TRS.

Assign each rule a number (ties allowed).

Label each step with the number of the applied rule.

R is confluent if every every critical peak admits a decreasing diagram.

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Gramlich–Lucas, Example 2

(1) $\mathsf{nats} \to 0 : \mathsf{inc}(\mathsf{nats})$ (2) $\mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)$ (3) $\mathsf{hd}(x : y) \to x$

(4) $\mathsf{tl}(x : y) \to y$ (5) $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$

Only one critical peak:

$$\mathsf{tl}(\mathsf{inc}(\mathsf{nats}))\ {}_5\!\leftarrow\ \mathsf{inc}(\mathsf{tl}(\mathsf{nats}))\ \to_1\ \mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats})))$$

with joins:

$$\to_1 \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nats}))) \to_2 \mathsf{tl}(\mathsf{s}(0) : \mathsf{s}(\mathsf{inc}(\mathsf{nats}))) \to_4\ \mathsf{inc}(\mathsf{inc}(\mathsf{nats})) \leftarrow_4$$