ISR 2026 — Basic Track

Jörg Endrullis

Cynthia Kop

Femke van Raamsdonk

partially based on slides by: Aart Middeldorp

hosting institution:
Radboud Universiteit Nijmegen
The Netherlands

- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Confluence related properties

we have seen:

CR $\Rightarrow$ WCR but WCR $\not\Rightarrow$ CR

CR $\Rightarrow$ NF $\Rightarrow$ UN $\Rightarrow$ UN$^{\rightarrow}$

in a confluent system, a term has at most one normal form

so confluence is nice to have

Outline

Example (NF)

Definition NF:

$\forall a, b \quad$ if $a \leftrightarrow^* b$ and b is a normal form then $a \to^* b$.

Example (NF)

Definition NF:

$\forall a, b$ if $a \leftrightarrow^* b$ and b is a normal form then $a \to^* b$.

TRS $\mathcal{R}$:

$$\mathrm{f}(x, x) \to n$$

The TRS $\mathcal{R}$ has the property NF. Why?

Example (NF)

Definition NF:

$\forall a, b$ if $a \leftrightarrow^* b$ and b is a normal form then $a \rightarrow^* b$.

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \rightarrow n$$

The TRS $\mathcal{R}$ has the property NF. Why?

TRS $\mathcal{S}$:

$$\begin{tikzcd} & \mathsf{a} \arrow[dl] \arrow[dr] & \\ \mathsf{b} \arrow[loop left] & & \mathsf{c} \arrow[loop right] \end{tikzcd}$$

The TRS $\mathcal{S}$ has the property NF. Why?

Example (NF)

Definition NF:

$\forall a, b$ if $a \leftrightarrow^* b$ and b is a normal form then $a \to^* b$.

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to n$$

The TRS $\mathcal{R}$ has the property NF. Why?

TRS $\mathcal{S}$:

$$\begin{tikzcd} & \mathsf{a} \arrow[dl] \arrow[dr] & \\ \mathsf{b} \arrow[loop left] & & \mathsf{c} \arrow[loop right] \end{tikzcd}$$

The TRS $\mathcal{S}$ has the property NF. Why?

The combination $\mathcal{R} \oplus \mathcal{S}$ does not have the property NF:

$$\mathsf{f}(\mathsf{b}, \mathsf{c}) \leftarrow \mathsf{f}(\mathsf{a}, \mathsf{c}) \leftarrow \mathsf{f}(\mathsf{a}, \mathsf{a}) \to n$$

By Aart Middeldorp.

Example (UN)

Definition UN:

$\forall a, b \quad$ if $a \leftrightarrow^* b$ and a, b are normal forms then $a = b$

Example (UN)

Definition UN:

$\forall a, b$ if $a \leftrightarrow^* b$ and a, b are normal forms then $a = b$

TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathsf{f}(x) & \to & x \\ \mathsf{f}(x) & \to & 0 \\ 0 & \to & 0 \end{array}$$

Is $\mathcal{R}$ CR?

Is $\mathcal{R}$ UN?

Example (UN)

Definition UN:

$\forall a, b \quad$ if $a \leftrightarrow^* b$ and a, b are normal forms then $a = b$

TRS $\mathcal{R}$:

$$\begin{aligned} \mathsf{f}(x) &\rightarrow x \\ \mathsf{f}(x) &\rightarrow 0 \\ 0 &\rightarrow 0 \end{aligned}$$

Is $\mathcal{R}$ CR?

Is $\mathcal{R}$ UN?

No:

$$y \leftarrow \mathsf{f}(y) \rightarrow 0 \leftarrow \mathsf{f}(x) \rightarrow x$$

Example ($\mathrm{UN}^{\rightarrow}$)

Definition $\mathrm{UN}^{\rightarrow}$:

$\forall a, b, c \quad$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$

Example ($\mathrm{UN}^{\rightarrow}$)

Definition $\mathrm{UN}^{\rightarrow}$:

$\forall a, b, c$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \rightarrow n$$

The TRS $\mathcal{R}$ has the property $\mathrm{UN}^{\rightarrow}$. Why?

Example ($\mathsf{UN}^{\rightarrow}$)

Definition $\mathsf{UN}^{\rightarrow}$:

$\forall a, b, c \quad$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \rightarrow n$$

The TRS $\mathcal{R}$ has the property $\mathsf{UN}^{\rightarrow}$. Why?

TRS $\mathcal{S}$:

$$\begin{array}{ccc} \mathsf{a} & \rightarrow & \mathsf{b} \\ \mathsf{a} & \rightarrow & \mathsf{c} \\ \mathsf{c} & \rightarrow & \mathsf{c} \\ \mathsf{d} & \rightarrow & \mathsf{c} \\ \mathsf{d} & \rightarrow & \mathsf{e} \end{array}$$

The TRS $\mathcal{S}$ has the property $\mathsf{UN}^{\rightarrow}$. Why?

Example ($\mathrm{UN}^{\rightarrow}$)

Definition $\mathrm{UN}^{\rightarrow}$:

$\forall a, b, c \quad$ if $a \rightarrow^{!} b$ and $a \rightarrow^{!} c$ then $b = c$

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \rightarrow n$$

The TRS $\mathcal{R}$ has the property $\mathrm{UN}^{\rightarrow}$. Why?

TRS $\mathcal{S}$:

$$\begin{array}{lcl} \mathsf{a} & \rightarrow & \mathsf{b} \\ \mathsf{a} & \rightarrow & \mathsf{c} \\ \mathsf{c} & \rightarrow & \mathsf{c} \\ \mathsf{d} & \rightarrow & \mathsf{c} \\ \mathsf{d} & \rightarrow & \mathsf{e} \end{array}$$

The TRS $\mathcal{S}$ has the property $\mathrm{UN}^{\rightarrow}$. Why?

The combination $\mathcal{R} \oplus \mathcal{S}$ does not have the property $\mathrm{UN}^{\rightarrow}$.

$$\mathsf{f}(\mathsf{a}, \mathsf{d}) \rightarrow \mathsf{f}(\mathsf{b}, \mathsf{d}) \rightarrow \mathsf{f}(\mathsf{b}, \mathsf{e})$$

Outline

Examples (Non-confluence)

$$
\begin{aligned}
\mathsf{a} &\rightarrow \mathsf{b} \\
\mathsf{a} &\rightarrow \mathsf{c}
\end{aligned}
$$

not confluent: $\mathsf{b} \leftarrow \mathsf{a} \rightarrow \mathsf{c}$

Examples (Non-confluence)

$$\begin{array}{ccc} \mathsf{a} & \to & \mathsf{b} \\ \mathsf{a} & \to & \mathsf{c} \end{array}$$

not confluent: $\mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$$

not confluent: $\mathsf{f}(\mathsf{a}) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \to \mathsf{a}$

Examples (More non-confluence)

$$\begin{array}{rcl} \mathsf{or}(x, y) & \to & x \\ \mathsf{or}(x, y) & \to & y \end{array}$$

not confluent: $x \leftarrow \mathsf{or}(x, y) \to y$

Examples (More non-confluence)

$$\begin{aligned} \mathrm{or}(x, y) &\rightarrow x \\ \mathrm{or}(x, y) &\rightarrow y \end{aligned}$$

not confluent: $x \leftarrow \mathrm{or}(x, y) \rightarrow y$

$$\begin{aligned} \mathrm{a} &\rightarrow \mathrm{b} \\ \mathrm{f}(\mathrm{a}) &\rightarrow \mathrm{c} \end{aligned}$$

not confluent: $\mathrm{f}(\mathrm{b}) \leftarrow \mathrm{f}(\mathrm{a}) \rightarrow \mathrm{c}$

Example

(Klop 1978)

$$\begin{array}{rcl} \mathsf{f}(x,x) & \to & \mathsf{a} \\ \mathsf{g}(x) & \to & \mathsf{f}(x,\mathsf{g}(x)) \\ \mathsf{c} & \to & \mathsf{g}(\mathsf{c}) \end{array}$$

not confluent:

$$\mathsf{c} \to \mathsf{g}(\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{f}(\mathsf{f}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \to \mathsf{a}$$

$$\mathsf{c} \to \mathsf{g}(\mathsf{c}) \to \mathsf{g}(\mathsf{g}(\mathsf{c})) \to^{*} \mathsf{g}(\mathsf{a})$$

Example

(Klop 1978)

$$\begin{aligned} \mathsf{f}(x,x) &\rightarrow \mathsf{a} \\ \mathsf{g}(x) &\rightarrow \mathsf{f}(x,\mathsf{g}(x)) \\ \mathsf{c} &\rightarrow \mathsf{g}(\mathsf{c}) \end{aligned}$$

not confluent:

$$\mathsf{c} \rightarrow \mathsf{g}(\mathsf{c}) \rightarrow \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \rightarrow \mathsf{f}(\mathsf{f}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \rightarrow \mathsf{a}$$

$$\mathsf{c} \rightarrow \mathsf{g}(\mathsf{c}) \rightarrow \mathsf{g}(\mathsf{g}(\mathsf{c})) \rightarrow^{*} \mathsf{g}(\mathsf{a})$$

(Huet 1980)

$$\begin{aligned} \mathsf{f}(x,x) &\rightarrow \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\rightarrow \mathsf{b} \\ \mathsf{c} &\rightarrow \mathsf{g}(\mathsf{c}) \end{aligned}$$

not confluent:

$$\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \rightarrow \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \rightarrow \mathsf{b}$$

Outline

how to prove CR?

for terminating systems, we can use Newman's Lemma:

SN and WCR $\Rightarrow$ CR

so: try to prove WCR, that seems easier

we will see: for terminating systems we can decide CR

for non-terminating systems: orthogonality on day 5

parallel steps: example

TRS $\mathcal{R}$:

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{c} &\to \mathsf{d} \end{aligned}$$

diverging parallel steps (the redexes occur at parallel positions):

$$\begin{tikzcd} & \mathsf{f}(\mathsf{a},\mathsf{c}) \arrow[dl] \arrow[dr] & \\ \mathsf{f}(\mathsf{b},\mathsf{c}) \arrow[dr, dotted] & & \mathsf{f}(\mathsf{a},\mathsf{d}) \arrow[dl, dotted] \\ & \mathsf{f}(\mathsf{b},\mathsf{d}) & \end{tikzcd}$$

we use left and right exactly one step to close the diagram

parallel steps: general

a one-step divergence with parallel steps can be joined

nested steps: example

TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathsf{f}(x) & \to & \mathsf{g}(x) \\ \mathsf{a} & \to & \mathsf{b} \end{array}$$

diverging nested steps (the redexes occur at nested positions):

$$\begin{tikzcd}
 & \mathsf{f}(\mathsf{a}) \arrow[dl] \arrow[dr] & \\
\mathsf{g}(\mathsf{a}) \arrow[dr, dotted] & & \mathsf{f}(\mathsf{b}) \arrow[dl, dotted] \\
 & \mathsf{g}(\mathsf{b}) &
\end{tikzcd}$$

we use left and right one step to close the diagram

this was just good luck

nested steps: example

TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathsf{f}(x) & \to & \mathsf{g}(x,x) \\ \mathsf{a} & \to & \mathsf{b} \end{array}$$

diverging nested steps (the redexes occur at nested positions):

$$\begin{tikzcd} & \mathsf{f}(\mathsf{a}) \arrow[dl] \arrow[dr] & \\ \mathsf{g}(\mathsf{a},\mathsf{a}) \arrow[dr, dotted, "*"] & & \mathsf{f}(\mathsf{b}) \arrow[dl, dotted] \\ & \mathsf{g}(\mathsf{b},\mathsf{b}) & \end{tikzcd}$$

on the left we need two steps because the redex a is copied

nested steps: example

TRS $\mathcal{R}$:

$$
\begin{aligned}
\mathsf{f}(x) &\to \mathsf{c} \\
\mathsf{a} &\to \mathsf{b}
\end{aligned}
$$

diverging nested steps (the redexes occur at nested positions):

$$
\begin{tikzcd}
& \color{blue}{\mathsf{f}}(\color{red}{\mathsf{a}}) \arrow[dl] \arrow[dr] & \\
\color{blue}{\mathsf{c}} \arrow[dr, dotted, two heads] & & \color{blue}{\mathsf{f}}(\color{red}{\mathsf{b}}) \arrow[dl, dotted, two heads] \\
& \mathsf{c} &
\end{tikzcd}
$$

on the left we need zero steps because the redex a is erased

nested steps: example

TRS $\mathcal{R}$:

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{g}(x) \\ \mathsf{a} &\to \mathsf{b} \end{aligned}$$

diverging nested steps (the redexes occur at nested positions):

$$\begin{tikzcd} & \mathsf{f}(\mathsf{a},\mathsf{a}) \arrow[dl] \arrow[dr] & \\ \mathsf{f}(\mathsf{b},\mathsf{a}) \arrow[dr, dotted, two heads] & & \mathsf{g}(\mathsf{a}) \arrow[dl, dotted, two heads] \\ & \mathsf{g}(\mathsf{b}) & \end{tikzcd}$$

we need to repair the equality of the two arguments of f

nested steps: general

we may have to deal with multiplication of redexes

(so erasure, duplication, triplication, ...)

we may have to repair the destruction of equality

but we can show WCR for diverging nested steps

as long as the patterns of the redexes do not overlap

overlap

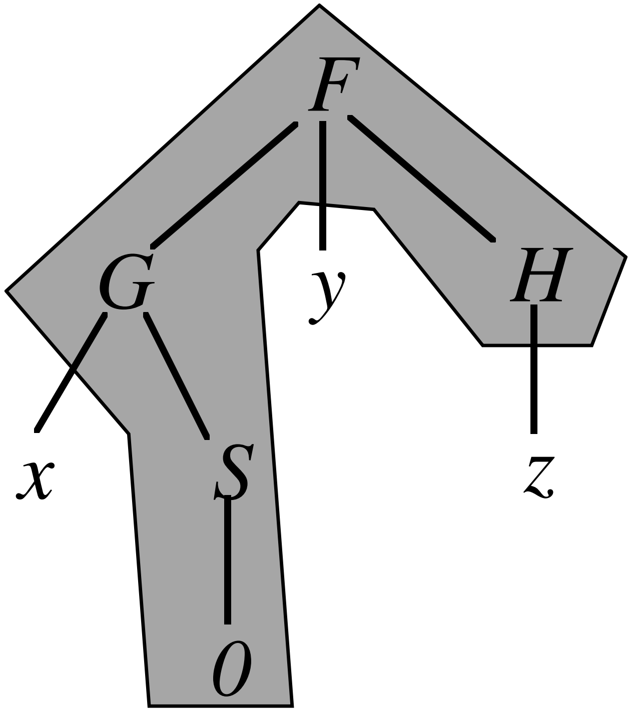
two redexes overlap

if there is at least one symbol that they both use in their pattern

we have seen overlap in some examples

Pattern

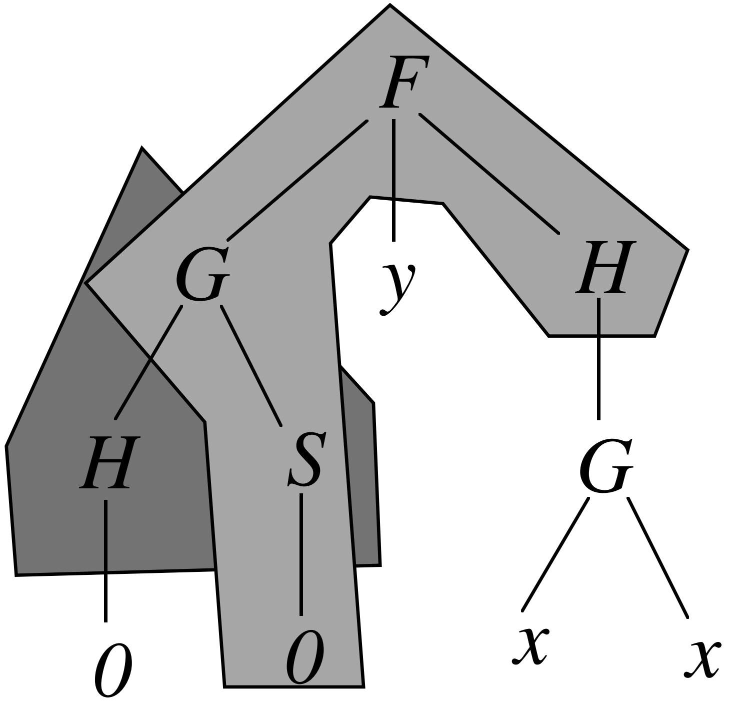
Pattern of a redex or rewrite rule:



$$F(G(x, S(0)), y, H(z)) \to G(x, z)$$

Overlap

Overlapping redex occurrences:



$$
\begin{array}{lllll}
\rho_1 : & F(G(x, S(0)), y, H(z)) & \to & x \\
\rho_2 : & G(H(x), S(y)) & \to & y
\end{array}
$$

overlapping steps

TRS $\mathcal{R}$:

$$
\begin{aligned}
\mathsf{f}(\mathsf{g}(x), y) &\to \mathsf{f'}(x, y) \\
\mathsf{g}(\mathsf{h}(z)) &\to \mathsf{g'}(z) \\
\mathsf{f}(\mathsf{g'}(x), y) &\to \mathsf{a} \\
\mathsf{f'}(x, y) &\to \mathsf{a}
\end{aligned}
$$

$$
\begin{tikzcd}
 & \mathsf{f}(\mathsf{g}(\mathsf{h}(x)), y) \arrow[dl] \arrow[dr] & \\
\mathsf{f}(\mathsf{g'}(x), y) \arrow[dr, dotted, two heads] & & \mathsf{f'}(\mathsf{h}(x), y) \arrow[dl, dotted, two heads] \\
 & \mathsf{a} &
\end{tikzcd}
$$

overlapping step: there is at least one symbol used in two different steps

overlapping steps: another one

TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathsf{f}(\mathsf{g}(x), y) & \to & \mathsf{f'}(x, y) \\ \mathsf{g}(\mathsf{h}(z)) & \to & \mathsf{g'}(z) \\ \mathsf{f}(\mathsf{g'}(x), y) & \to & \mathsf{a} \\ \mathsf{f'}(x, y) & \to & \mathsf{a} \end{array}$$

$$\begin{tikzcd} & \mathsf{f}({\color{red}\mathsf{g}}(\mathsf{h}(x)), {\color{blue}\mathsf{h}}(\mathsf{y})) \arrow[dl] \arrow[dr] & \\ \mathsf{f}(\mathsf{g'}(x), {\color{blue}\mathsf{h}}(y)) \arrow[dr, dotted, ->>] & & \mathsf{f'}(\mathsf{h}(x), {\color{blue}\mathsf{h}}(y)) \arrow[dl, dotted, ->>] \\ & \mathsf{a} & \end{tikzcd}$$

overlapping steps: yet another one

TRS $\mathcal{R}$:

$$
\begin{array}{rcl}
\mathsf{f}(\mathsf{g}(x), y) & \to & \mathsf{f'}(x, y) \\
\mathsf{g}(\mathsf{h}(z)) & \to & \mathsf{g'}(z) \\
\mathsf{f}(\mathsf{g'}(x), y) & \to & \mathsf{a} \\
\mathsf{f'}(x, y) & \to & \mathsf{a}
\end{array}
$$

$$
\begin{tikzcd}
& \mathsf{f}(\mathsf{g}(\mathsf{h}(\mathsf{h}(x))), y) \arrow[dl] \arrow[dr] & \\
\mathsf{f}(\mathsf{g'}(\mathsf{h}(\mathsf{h}(x))), y) \arrow[dr, dotted, twoheadrightarrow] & & \mathsf{f'}(\mathsf{g}(\mathsf{h}(\mathsf{h}(x))), y) \arrow[dl, dotted, twoheadrightarrow] \\
& \mathsf{a} &
\end{tikzcd}
$$

observation

the second and third divergences are instances of the first one

$$\mathsf{f}(\mathsf{g}'(x), y) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(x)), y) \rightarrow \mathsf{f}'(\mathsf{h}(x), y)$$

$$\mathsf{f}(\mathsf{g}'(x), \mathsf{h}(y)) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(x)), \mathsf{h}(y)) \rightarrow \mathsf{f}'(\mathsf{h}(x), \mathsf{h}(y))$$

$$\mathsf{f}(\mathsf{g}'(\mathsf{h}(x)), y) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(\mathsf{h}(x))), y) \rightarrow \mathsf{f}'(\mathsf{h}(\mathsf{h}(x)), y)$$

Outline

plan

we aim to prove WCR

we can deal with parallel steps

we can deal with nested non-overlapping steps

for overlapping steps, we consider the most general form

a condition on that most general form guarantees joinability

overlap and critical pairs

overlap:

we put a (small) left-hand side g ‘on’ a (large) left-hand l

we take this in the most general (variables where possible) form

critical pair

from the overlap we can go left with the small redex
and right with the large redex

this is an overlapping step in the most general form

the resulting pair is a critical pair

critical pair: example

$$
\begin{array}{rcl}
\mathsf{f}(\mathsf{g}(x), y) & \to & \mathsf{f'}(x, y) \\
\mathsf{g}(\mathsf{h}(z)) & \to & \mathsf{g'}(z) \\
\mathsf{f}(\mathsf{g'}(x), y) & \to & \mathsf{a} \\
\mathsf{f'}(x, y) & \to & \mathsf{a}
\end{array}
$$

critical pair:

$$\mathsf{f}(\mathsf{g'}(x), y) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(x)), y) \to \mathsf{f'}(\mathsf{h}(x), y)$$

the other overlapping steps are instances of our critical pair:

$$\mathsf{f}(\mathsf{g'}(x), \mathsf{h}(y)) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(x)), \mathsf{h}(y)) \to \mathsf{f'}(\mathsf{h}(x), \mathsf{h}(y))$$

$$\mathsf{f}(\mathsf{g'}(\mathsf{h}(x)), y) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{h}(\mathsf{h}(x))), y) \to \mathsf{f'}(\mathsf{h}(\mathsf{h}(x)), y)$$

back to the plan

we can compute critical pairs using *unification*

crucial: if all critical pairs are joinable,

then all overlapping steps are joinable

we already could join parallel steps and non-overlapping nested steps

this gives us an important result:

Critical Pair Lemma (Huet 1980)

A TRS is WCR if and only if all critical pairs are convergent.

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

We decide WCR as follows:

For every critical pair s and t we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\rightarrow(s) \cap \rightarrow(t) \neq \emptyset$. $\blacksquare$

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

We decide WCR as follows:

For every critical pair s and t we check $s \downarrow t$ by computing:

- the set of reducts of s: $\to(s) = \{s' \mid s \to^* s'\}$, and
- the set of reducts of t: $\to(t) = \{t' \mid t \to^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\to(s) \cap \to(t) \neq \emptyset$. $\blacksquare$

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

Proof.

For terminating R we have CR $\Leftrightarrow$ WCR.

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

Proof.

For terminating R we have CR $\Leftrightarrow$ WCR. We decide WCR as follows:

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

Proof.

For terminating R we have CR $\Leftrightarrow$ WCR. We decide WCR as follows:

For every critical pair s and t we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Local Confluence

Critical Pair Lemma

A TRS is WCR $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

Proof.

For terminating R we have CR $\Leftrightarrow$ WCR. We decide WCR as follows:

For every critical pair s and t we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\rightarrow(s) \cap \rightarrow(t) \neq \emptyset$. ∎