ISR 2026 — Basic Track

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partially based on slides by: Aart Middeldorp

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- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

Outline

Question: how long does this take?

$$
\begin{array}{rcl}
\mathsf{add}(\mathsf{0}, y) & \rightarrow & y \\
\mathsf{add}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{add}(x, y)) \\
\mathsf{mul}(\mathsf{0}, y) & \rightarrow & \mathsf{0} \\
\mathsf{mul}(\mathsf{s}(x), y) & \rightarrow & \mathsf{add}(y, \mathsf{mul}(x, y))
\end{array}
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Challenge: how many steps does it take (worst-case) to reduce $\mathsf{mul}(\mathsf{s}^n(0), \mathsf{s}^m(0))$ to normal form?

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Challenge: what if the rules are instead given as:

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Observation: interpretations to $\mathbb{N}$

Recall: if $s \to_{\mathcal{R}} t$ then $[\![s]\!] > [\![t]\!]$

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$\Longrightarrow$ $[\![s]\!]$ bounds the reduction length!

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[0] &= 0 \\
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- we only need *order* of magnitude anyway
- this problem is solved by using cost-size interpretations
- better techniques are also available

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Definition

The derivational complexity of a TRS $\mathcal{R}$ is a function dc from $\mathbb{N}$ to $\mathbb{N}$, such that:

$$\begin{array}{c} \text{if } |s| \leq n \\ \text{then derivationheight}(s) \leq dc(n) \end{array}$$

Derivational complexity example

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$$

Challenge: what is the derivational complexity of this TRS?

Derivational complexity explosion?

$$\begin{array}{rcl} \mathsf{add}(0, y) & \to & y \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \end{array}$$

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Runtime complexity

Two kinds of symbols

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A basic term is a term $\mathsf{f}(s_1, \ldots, s_n)$ where f is a defined symbol and all s_i are ground constructor terms.

Definition

The runtime complexity of a TRS $\mathcal{R}$ is a function rc from $\mathbb{N}$ to $\mathbb{N}$, such that:

if $|s| \leq n$
and s is a basic term
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$$\begin{aligned}
\mathsf{minus}(x, \mathsf{0}) &\to x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{minus}(x, y) \\
\mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) &\to \mathsf{0} \\
\mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y)))
\end{aligned}$$

Proving runtime complexity using interpretations to $\mathbb{N}$

Suppose constructors c have an interpretation $[\mathsf{c}](x_1, \ldots, x_n) = x_1 + \cdots + x_n + b$.

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Challenge: if $[f]$ is a linear polynomial, can we bound $[\![s]\!]$ for basic s of size $\leq n$?

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Suppose constructors c have an interpretation $[\mathsf{c}](x_1, \ldots, x_n) = x_1 + \cdots + x_n + b$.

Challenge: what can $[\![s]\!]$ be if s is a ground constructor term of size $|s| \leq n$?

Challenge: if $[f]$ is a cubic polynomial, can we bound $[\![s]\!]$ for basic s of size $\leq n$?

Other challenge: and what if $[\mathsf{c}](x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n + b$?