ISR 2026 — Basic Track

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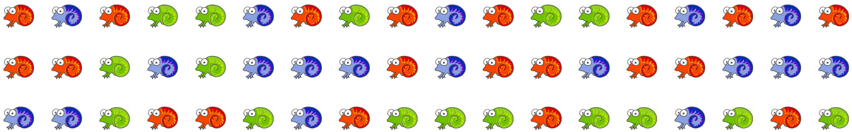
partially based on slides by: Aart Middeldorp

hosting institution:
Radboud Universiteit Nijmegen
The Netherlands

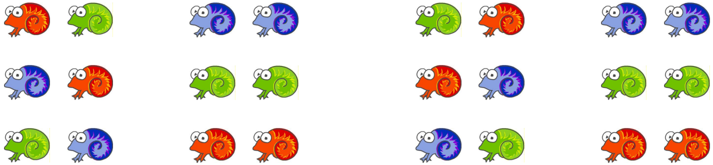
- Part 1-2: Abstract Rewriting
- Part 3: Term Rewriting
- Part 5: Termination: Monotonic Algebras
- Part 6: Termination: Recursive Path Ordering
- Part 7: Confluence: Unification and Critical Pairs
- Part 9-10: Weak and Strong Normalization for Typed Lambda Calculus
- Part 11: Equational Reasoning and Completion
- Part 13: Termination: Dependency Pair Framework
- Part 15: Confluence: Orthogonality
- Part 16: Confluence: Decreasing Diagrams
- Part 17: Advanced Topics

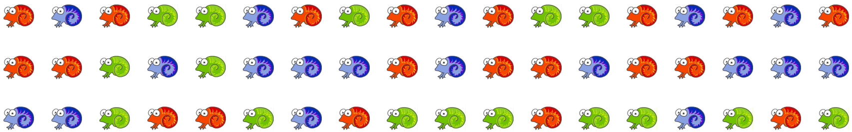
Outline

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?





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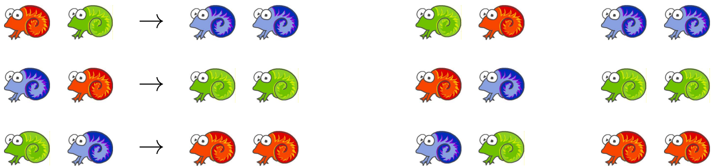

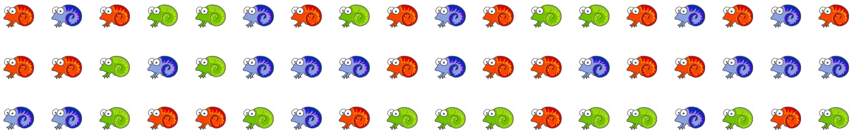

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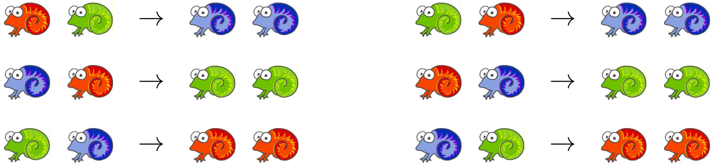

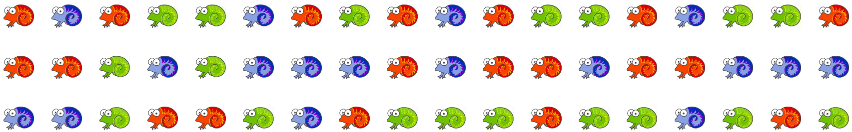

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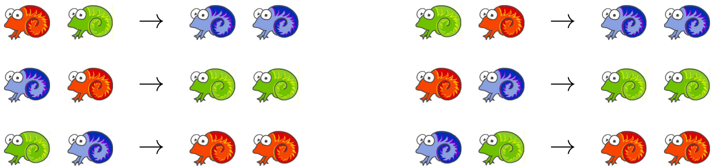

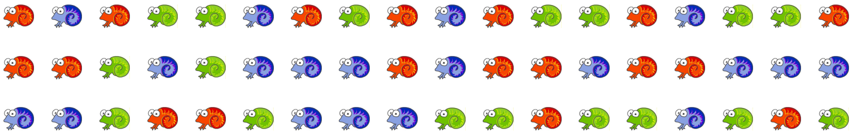

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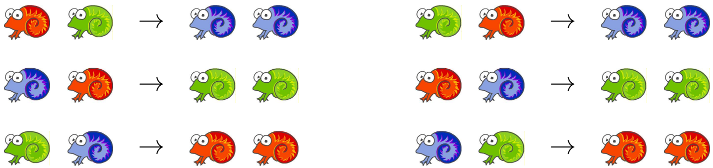

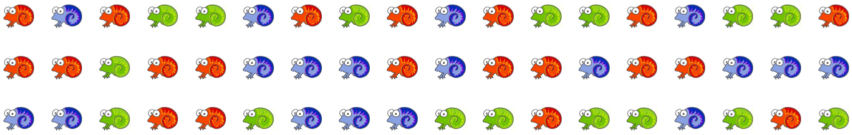

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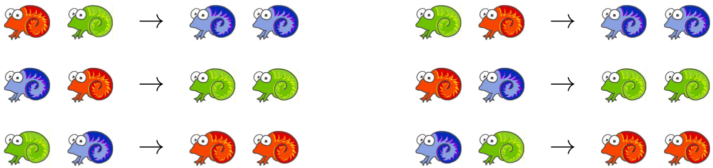

A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

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Techniques exist to perform the following DNA substitutions

$$\mathsf{TCAT} \leftrightarrow \mathsf{T} \quad \mathsf{GAG} \leftrightarrow \mathsf{AG} \quad \mathsf{CTC} \leftrightarrow \mathsf{TC} \quad \mathsf{AGTA} \leftrightarrow \mathsf{A} \quad \mathsf{TAT} \leftrightarrow \mathsf{CT}$$

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Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T   GAG $\leftrightarrow$ AG   CTC $\leftrightarrow$ TC   AGTA $\leftrightarrow$ A   TAT $\leftrightarrow$ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature $\quad 0$ (constants) $\quad s$ (unary) $\quad +$ (binary, infix)

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Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) $+$ (binary, infix)

terms $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x)+y$

rewrite rules

$$0 + y \rightarrow y$$
$$\mathsf{s}(x) + y \rightarrow \mathsf{s}(x + y)$$

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signature	0 (constants) $\quad \mathsf{s}$ (unary) $\quad +$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(x) + y$
rewrite rules	$0 + y \to y$
	$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
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rewriting $\quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \qquad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

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rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \qquad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

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rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \qquad y \mapsto \mathsf{s}(\mathsf{s}(0))$ $\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

Example (Group Theory)

signature $\quad e$ (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

Example (Group Theory)

signature	e	(constant)	$^-$	(unary, postfix)	$\cdot$	(binary, infix)	
equations	$\mathsf{e} \cdot x \approx x$		$x^- \cdot x \approx \mathsf{e}$		$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$

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equations $\quad \mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

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$$\mathsf{e}^- =$$

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$$\mathsf{e}^- = \mathsf{e} * \mathsf{e}^- =$$

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$$\mathsf{e}^- = \mathsf{e} * \mathsf{e}^- = \mathsf{e} * (\mathsf{e} * \mathsf{e}^-) =$$

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signature	e	(constant)	$^{-}$	(unary, postfix)	$\cdot$	(binary, infix)	
equations	$\mathsf{e} \cdot x \approx x$		$x^{-} \cdot x \approx \mathsf{e}$		$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$
theorems		$\mathsf{e}^{-} \approx_{\mathcal{E}} \mathsf{e}$		$(x \cdot y)^{-} \approx_{\mathcal{E}} y^{-} \cdot x^{-}$			

$$
\begin{aligned}
\mathsf{e}^{-} &= \mathsf{e} * \mathsf{e}^{-} = \mathsf{e} * (\mathsf{e} * \mathsf{e}^{-}) = ((\mathsf{e}^{-})^{-} * \mathsf{e}^{-}) * (\mathsf{e} * \mathsf{e}^{-}) \\
&=
\end{aligned}
$$

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theorems	$\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e}$	$(x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot x^-$		

$$\begin{aligned} \mathsf{e}^- &= \mathsf{e} * \mathsf{e}^- = \mathsf{e} * (\mathsf{e} * \mathsf{e}^-) = ((\mathsf{e}^-)^- * \mathsf{e}^-) * (\mathsf{e} * \mathsf{e}^-) \\ &= (\mathsf{e}^-)^- * ((\mathsf{e}^- * \mathsf{e}) * \mathsf{e}^-) = \end{aligned}$$

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\end{aligned}
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rewrite rules $\qquad R$

$$
\begin{aligned}
\mathsf{e} \cdot x &\to x \\
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\end{aligned}
$$

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rewrite rules			R

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\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
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(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^- & \to & \mathsf{e} & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & x \cdot (x^- \cdot y) & \to & y
\end{array}$$

$$\begin{aligned}
\mathsf{e}^- &= \mathsf{e} * \mathsf{e}^- = \mathsf{e} * (\mathsf{e} * \mathsf{e}^-) = ((\mathsf{e}^-)^- * \mathsf{e}^-) * (\mathsf{e} * \mathsf{e}^-) \\
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\end{array}$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

$$\begin{aligned}
\mathrm{e}^- &= \mathrm{e} * \mathrm{e}^- = \mathrm{e} * (\mathrm{e} * \mathrm{e}^-) = ((\mathrm{e}^-)^- * \mathrm{e}^-) * (\mathrm{e} * \mathrm{e}^-) \\
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signature	e	(constant)	$^-$	(unary, postfix)	$\cdot$	(binary, infix)	
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rewrite rules

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\mathsf{e}^- & \to & \mathsf{e} & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & x \cdot (x^- \cdot y) & \to & y
\end{array}
\qquad R
$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

② R admits no infinite computations

$$
\begin{aligned}
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\end{aligned}
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\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
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(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\
x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y
\end{array}
$$

① $\quad s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

② $\quad R$ admits no infinite computations

① & ② $\quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem

$$
\begin{aligned}
\mathsf{e}^{-} &= \mathsf{e} * \mathsf{e}^{-} = \mathsf{e} * (\mathsf{e} * \mathsf{e}^{-}) = ((\mathsf{e}^{-})^{-} * \mathsf{e}^{-}) * (\mathsf{e} * \mathsf{e}^{-}) \\
&= (\mathsf{e}^{-})^{-} * ((\mathsf{e}^{-} * \mathsf{e}) * \mathsf{e}^{-}) = (\mathsf{e}^{-})^{-} * (\mathsf{e} * \mathsf{e}^{-}) = (\mathsf{e}^{-})^{-} * \mathsf{e}^{-} = \mathsf{e}
\end{aligned}
$$

Example (Combinatory Logic)

signature $\quad$ S K I (constants)

Example (Combinatory Logic)

signature $\quad \mathsf{S} \quad \mathsf{K} \quad \mathsf{I}$ (constants) $\quad \cdot$ (application, binary, infix)

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K}\cdot\mathsf{I})\cdot\mathsf{I})\cdot\mathsf{S}$ $(x\cdot z)\cdot(y\cdot z)$

rewrite rules

$$\begin{aligned} \mathsf{I}\cdot x &\to x \\ (\mathsf{K}\cdot x)\cdot y &\to x \\ ((\mathsf{S}\cdot x)\cdot y)\cdot z &\to (x\cdot z)\cdot(y\cdot z) \end{aligned}$$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $\quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $\quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$
\begin{aligned}
\mathsf{I} \cdot x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z)
\end{aligned}
$$

rewriting $((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x$

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rewriting $((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x \ \to\ (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x)$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

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rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

Example (Combinatory Logic)

signature $\mathsf{S} \quad \mathsf{K} \quad \mathsf{I}$ (constants) $\quad \cdot \quad$ (application, binary, infix)

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rewrite rules

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

inventor Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature λ (binds variables)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

Example (Lambda Calculus)

signature	λ (binds variables) $\cdot$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$
β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

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replace free occurrences of x in M by N

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	$(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)$

Example (Lambda Calculus)

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replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\rightarrow\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)$$
$$\rightarrow \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z)$$

Example (Lambda Calculus)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

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rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\rightarrow\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$
\begin{aligned}
&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a) \\
&\rightarrow \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z) \\
&\rightarrow \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z)
\end{aligned}
$$

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rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$
\begin{aligned}
&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a) \\
&\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z) \\
&\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z
\end{aligned}
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Example (Lambda Calculus)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$\begin{aligned}&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)\\ &\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z)\\ &\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z\end{aligned}$$

$(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z)$

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rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$
\begin{aligned}
&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a) \\
&\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z) \\
&\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z
\end{aligned}
$$

$$(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z) \to (\lambda y.\, \lambda z.\, y \cdot z) \cdot (\lambda y.\, \lambda z.\, y \cdot z)$$

Example (Lambda Calculus)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

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replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$
\begin{aligned}
&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)\\
&\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z)\\
&\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z
\end{aligned}
$$

$$
\begin{aligned}
&(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z) \to (\lambda y.\, \lambda z.\, y \cdot z) \cdot (\lambda y.\, \lambda z.\, y \cdot z)\\
&\to \lambda z.\, (\lambda y.\, \lambda z.\, y \cdot z) \cdot z
\end{aligned}
$$

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$\begin{aligned}&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)\\ &\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z)\\ &\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z\end{aligned}$$

$$\begin{aligned}&(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z) \to (\lambda y.\, \lambda z.\, y \cdot z) \cdot (\lambda y.\, \lambda z.\, y \cdot z)\\ &\to \lambda z.\, (\lambda y.\, \lambda z.\, y \cdot z) \cdot z\end{aligned}$$

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

$$\begin{aligned}&(\lambda x.\lambda y.\lambda z.\, (x \cdot z) \cdot (y \cdot z)) \cdot (\lambda a.\lambda b.\, a)\\ &\to \lambda y.\lambda z.\, ((\lambda a.\lambda b.\, a) \cdot z) \cdot (y \cdot z)\\ &\to \lambda y.\lambda z.\, (\lambda b.\, z) \cdot (y \cdot z) \to \lambda y.\lambda z.\, z\end{aligned}$$

$$\begin{aligned}&(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z) \to (\lambda y.\, \lambda z.\, y \cdot z) \cdot (\lambda y.\, \lambda z.\, y \cdot z)\\ &\to \lambda z.\, (\lambda y.\, \lambda z.\, y \cdot z) \cdot z \to \lambda z.\, \lambda w.\, z \cdot w\end{aligned}$$

Example (Lambda Calculus)

signature	λ (binds variables) $\quad\cdot\quad$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
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	$(\lambda x.\, x \cdot x) \cdot (\lambda y.\, \lambda z.\, y \cdot z) \to (\lambda y.\, \lambda z.\, y \cdot z) \cdot (\lambda y.\, \lambda z.\, y \cdot z)$ $\to \lambda z.\, (\lambda y.\, \lambda z.\, y \cdot z) \cdot z \to \lambda z.\, \lambda w.\, z \cdot w$
inventor	Alonzo Church (1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid \lambda x.\, M \mid M\, M$

α conversion $\lambda x.\, x\, y =_\alpha \lambda z.\, z\, y$

β reduction $(\lambda x.\, M)\, N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x\, x)\, \lambda x.\, x\, x \;\to\; (\lambda x.\, x\, x)\, \lambda x.\, x\, x$

$(\lambda xyz.\, x\, z\, (y\, z))\, \lambda ab.\, a$
$\to \lambda yz.\, (\lambda ab.\, a)\, z\, (y\, z)$
$\to \lambda yz.\, (\lambda b.\, z)\, (y\, z) \to \lambda yz.\, z$

$(\lambda x.\, x\, x)\, \lambda yz.\, y\, z \to (\lambda yz.\, y\, z)\, \lambda yz.\, y\, z$
$\to \lambda z.\, (\lambda y z.\, y\, z)\, z \to \lambda zw.\, z\, w$

inventor Alonzo Church (1936)

Motivation

Term rewriting is used in:

- functional programming (higher order term rewriting)
- model checking (e.g. mCRL)
- compiler construction (graph rewriting)
- computer algebra systems (e.g. Mathematica, Wolfram Alpha)
- proof assistants / automated theorem provers
- deciding equality in equational systems (axiom systems)
- abstract model of computation
- ...

Outline

Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

- string rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- $\ldots$

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

Definitions

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$$
\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
 & e \arrow[u] \arrow[dr] & f \arrow[l] \arrow[d] & \\
 & & g &
\end{tikzcd}
$$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
 & \mathsf{e} \arrow[u] \arrow[dr] & \mathsf{f} \arrow[l] \arrow[d] & \\
 & & \mathsf{g} &
\end{tikzcd}$$

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$
- $\to = \left\{\begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix}\right\}$

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\end{tikzcd}
$$

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to = \left\{ \begin{matrix} (a, e), (b, a), (b, c), (c, d), (c, f) \\ (e, b), (e, g), (f, e), (f, g) \end{matrix} \right\}$

- rewrite sequence
  - finite $\quad a \to e \to b \to c \to f$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

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\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
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 & & \mathsf{g} &
\end{tikzcd}$$

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- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$
- $\to = \left\{ \begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix} \right\}$

- rewrite sequence

rewrite step

  - finite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$
\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
& e \arrow[u] \arrow[dr] & f \arrow[l] \arrow[d] & \\
& & g &
\end{tikzcd}
$$

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to = \left\{ \begin{matrix} (a,e), (b,a), (b,c), (c,d), (c,f) \\ (e,b), (e,g), (f,e), (f,g) \end{matrix} \right\}$

- rewrite sequence
  - finite $\quad a \to e \to b \to c \to f$ (rewrite step)
  - empty $\quad a$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
& \mathsf{e} \arrow[u] \arrow[dr] & \mathsf{f} \arrow[l] \arrow[d] & \\
& & \mathsf{g} &
\end{tikzcd}$$

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- rewrite sequence

rewrite step

  - finite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$
  - empty $\quad \mathsf{a}$
  - infinite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \cdots$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
 & \mathsf{e} \arrow[u] \arrow[dr] & \mathsf{f} \arrow[l] \arrow[d] & \\
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\end{tikzcd}$$

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$
- $\rightarrow = \left\{ \begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix} \right\}$

- rewrite sequence

rewrite step

  - finite $\mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \mathsf{c} \rightarrow \mathsf{f}$ length 4
  - empty a length 0
  - infinite $\mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \cdots$ length ω

The length of a rewrite sequence is the number of rewrite steps.

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
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ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to = \left\{ \begin{matrix} (a,e),(b,a),(b,c),(c,d),(c,f) \\ (e,b),(e,g),(f,e),(f,g) \end{matrix} \right\}$

- rewrite sequence

rewrite step

  - finite $a \to e \to b \to c \to f$ length 4 $a \to^* f$
  - empty a length 0 $a \to^* a$
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The length of a rewrite sequence is the number of rewrite steps.
We write $x \to^* y$ if x rewrites to y in 0 or more steps.

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- $\leftarrow$ or $\to^{-1}$ inverse of $\to$

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- if $x \to^* y$ then x rewrites to y and y is reduct of x

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 & & g &
\end{tikzcd}$$

- $a \to^* f$

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- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\!\leftarrow y$ then z is common reduct of x and y

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Outline

Definitions

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  - no infinite rewrite sequences

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Lemmata

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$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$

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$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$

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$$
\begin{tikzcd}
 & a \arrow[dl, two heads] \arrow[dr, two heads] & \\
b \arrow[dr, dotted, two heads] & & c \arrow[dl, dotted, two heads] \\
 & d &
\end{tikzcd}
$$

in diagrams: $\twoheadrightarrow$ for $\to^*$

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    in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

    $\exists d$
  - $\forall a, b, c.\quad a \rightarrow^* b \;\wedge\; a \rightarrow^* c \quad\Rightarrow\quad \exists d.\quad b \rightarrow^* d \;\wedge\; c \rightarrow^* d$

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Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

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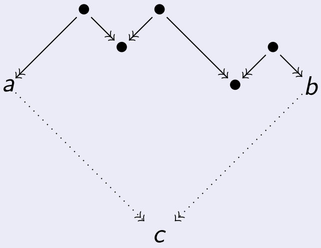

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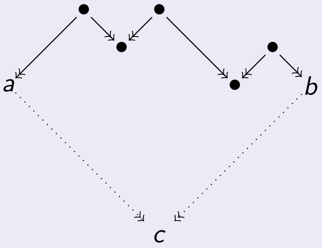

  $\exists c$

- $\forall a, b. \quad a \leftrightarrow^* b \quad \Rightarrow \quad \exists c. \quad a \rightarrow^* c \ \wedge \ b \rightarrow^* c$

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

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First proof: diagram tiling.

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- $\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.
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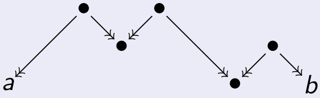

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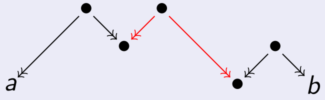

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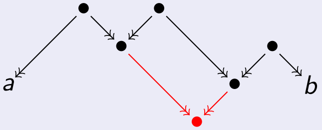

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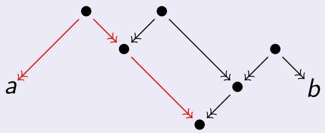

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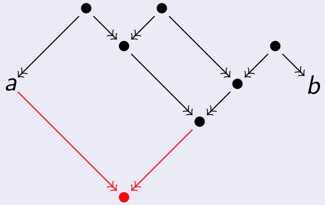

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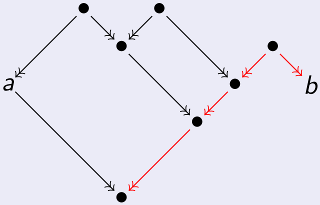

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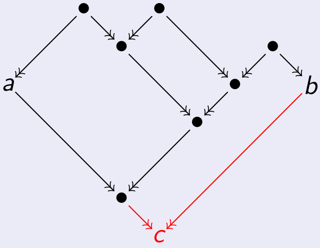

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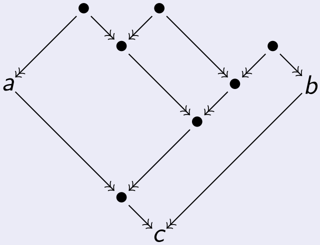

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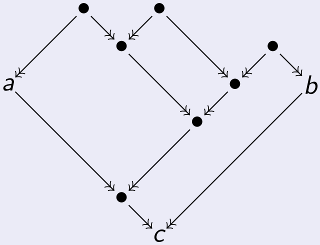

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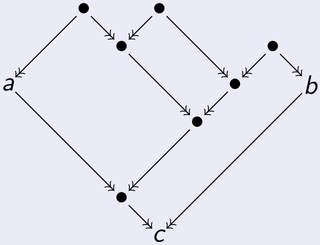

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Induction

Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

1. *The base case:*
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2. *The inductive step:*
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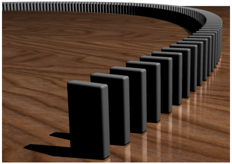

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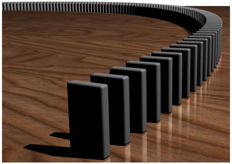

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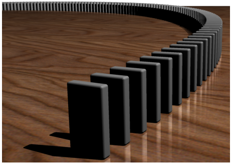

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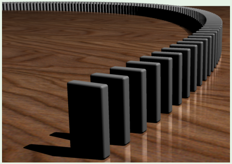

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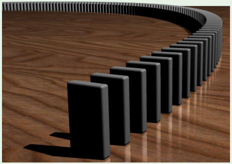

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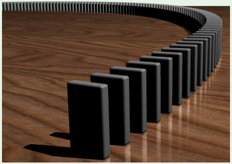

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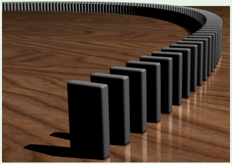

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- Base case: proof that the first domino falls
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Then you have proven that all dominoes will fall.

Example

We use induction to prove that:

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

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Hence the formula holds for all $n \in \mathbb{N}$.

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Second proof: induction.

- $\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.
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  We conclude $a \rightarrow^* \cdot {}^*\!\leftarrow b$, that is, $a \downarrow b$.

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  Let $a\,(\leftarrow^* \cdot \rightarrow^*)^{n+1}\, b$. Then $a\,(\leftarrow^* \cdot \rightarrow^*)^n\, d \leftarrow^* e \rightarrow^* b$ for some d, e.
  - Hence $a \rightarrow^* f\ {}^*\!\!\leftarrow d$ for some f by induction hypothesis.
  - Now $f \leftarrow^* e \rightarrow^* b$ and thus $f \rightarrow^* \cdot {}^*\!\!\leftarrow b$ since by assumption $\uparrow \subseteq \downarrow$.

  We conclude $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$, that is, $a \downarrow b$.

Hence we have shown $\uparrow^* \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

$\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We show $\leftrightarrow^* \subseteq \downarrow$.

We proof by induction on n that $a \, ({}^*\!\!\leftarrow \cdot \rightarrow^*)^n \, b$ implies $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$.

- Base case $n = 0$: $a \, ({}^*\!\!\leftarrow \cdot \rightarrow^*)^0 \, b$. Then $a = b$ and hence $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$.
- Induction step $n + 1$: (assume it holds for n, show it for $n + 1$)

  Let $a \, ({}^*\!\!\leftarrow \cdot \rightarrow^*)^{n+1} \, b$. Then $a \, ({}^*\!\!\leftarrow \cdot \rightarrow^*)^n \, d \, {}^*\!\!\leftarrow e \rightarrow^* b$ for some d, e.
  - Hence $a \rightarrow^* f \; {}^*\!\!\leftarrow d$ for some f by induction hypothesis.
  - Now $f \, {}^*\!\!\leftarrow e \rightarrow^* b$ and thus $f \rightarrow^* \cdot {}^*\!\!\leftarrow b$ since by assumption $\uparrow \subseteq \downarrow$.

  We conclude $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$, that is, $a \downarrow b$.

Hence we have shown $\uparrow^* \subseteq \downarrow$. It follows $\leftrightarrow^* \subseteq \downarrow$ since $\leftrightarrow^* \subseteq \uparrow^*$. ∎

Lemma

Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$

Lemma

Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
- $\forall a, b, c$

  $\exists d$

$$\begin{tikzcd} a \arrow[r, two heads] \arrow[d] & b \arrow[d, dotted, two heads] \\ c \arrow[r, dotted, two heads] & d \end{tikzcd}$$

Lemma

Confluence $\uparrow \ \subseteq \ \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \ \subseteq \ \downarrow$
- $\forall a, b, c$ $\exists d$
$$\begin{tikzcd} a \arrow[rr, twoheadrightarrow] \arrow[d] & & b \arrow[d, dotted, twoheadrightarrow] \\ c \arrow[rr, dotted, twoheadrightarrow] & & d \end{tikzcd}$$
- $\forall a, b, c. \quad a \rightarrow^* b \ \wedge \ a \rightarrow c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \ \wedge \ c \rightarrow^* d$

Lemma

$$\uparrow \;\subseteq\; \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \;\subseteq\; \downarrow$$

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

- $\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \to^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \to^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \to^* \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[rr, two heads] \arrow[d] & & b \\
{} \arrow[d] & & \\
{} \arrow[d] & & \\
{} \arrow[d] & & \\
{} \arrow[d] & & \\
c & &
\end{tikzcd}$$

Lemma

$$\uparrow \, \subseteq \, \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \, \subseteq \, \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \, \subseteq \, \downarrow$. We have $\leftarrow \cdot \rightarrow^* \, \subseteq \, \uparrow \, \subseteq \, \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \, \subseteq \, \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \, \subseteq \, \downarrow$. We show $\uparrow \, \subseteq \, \downarrow$.

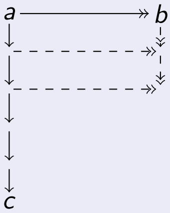
$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \\
{} \arrow[d] & \\
{} \arrow[d] & \\
{} \arrow[d] & \\
c &
\end{tikzcd}$$

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

- $\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.
- $\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



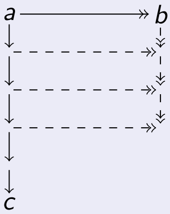
Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \\
c &
\end{tikzcd}$$

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & {}
\end{tikzcd}$$

Lemma

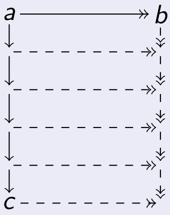
$$\uparrow\ \subseteq\ \downarrow \quad\Longleftrightarrow\quad \leftarrow \cdot \rightarrow^*\ \subseteq\ \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow\ \subseteq\ \downarrow$. We have $\leftarrow \cdot \rightarrow^*\ \subseteq\ \uparrow\ \subseteq\ \downarrow$. Hence $\leftarrow \cdot \rightarrow^*\ \subseteq\ \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^*\ \subseteq\ \downarrow$. We show $\uparrow\ \subseteq\ \downarrow$.

By induction on n we show: ${}^n\!\leftarrow \cdot \rightarrow^*\ \subseteq\ \downarrow$ for all n.



Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
\cdot \arrow[r, dashed, two heads] \arrow[d] & \cdot \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & \cdot
\end{tikzcd}$$

By induction on n we show: ${}^n\!\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^0\!\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.

Lemma

$$\uparrow\ \subseteq\ \downarrow \quad\Longleftrightarrow\quad \leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow\ \subseteq\ \downarrow$. We have $\leftarrow\cdot\rightarrow^*\ \subseteq\ \uparrow\ \subseteq\ \downarrow$. Hence $\leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$.

$\Leftarrow$ Assume $\leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$. We show $\uparrow\ \subseteq\ \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & {}
\end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow\cdot\rightarrow^*\ =\ \rightarrow^*\ \subseteq\ \downarrow$.
- Induction step $n + 1$:

Lemma

$$\uparrow \subseteq \downarrow \quad \iff \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd} a \arrow[r, twoheadrightarrow] \arrow[d, "n+1"] & b \\ c & \end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$: let $c \ {}^{n+1}\!\leftarrow a \rightarrow^* b$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd} a \arrow[r, ->>] \arrow[d, "n"'] & b \\ d \arrow[d] & \\ c & \end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n+1$: let $c \;{}^{n+1}\!\leftarrow a \rightarrow^* b$.

  Then $c \leftarrow d \;{}^{n}\!\leftarrow a \rightarrow^* b$ for some d, and:

Lemma

$$\uparrow \;\subseteq\; \downarrow \quad\Longleftrightarrow\quad \leftarrow \cdot \rightarrow^* \;\subseteq\; \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[r, ->>] \arrow[d, "n"] \arrow[dr, phantom, "\mathrm{IH}"] & b \arrow[d, dashed, ->>] \\
d \arrow[r, dashed, ->>] \arrow[d] & e \\
c &
\end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n+1$: let $c\ {}^{n+1}\!\leftarrow a \rightarrow^* b$.

  Then $c \leftarrow d\ {}^{n}\!\leftarrow a \rightarrow^* b$ for some d, and:

  - By induction hypothesis $d \rightarrow^* e\ {}^{*}\!\leftarrow b$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

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$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d, "n"] \arrow[dr, phantom, "\mathrm{IH}"] & b \arrow[d, dashed, two heads] \\
d \arrow[r, dashed, two heads] \arrow[d] & e \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & f
\end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

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  - By induction hypothesis $d \rightarrow^* e \ {}^*\!\leftarrow b$.
  - Then $c \rightarrow^* f \ {}^*\!\leftarrow e$ since $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d, "n"'] \arrow[dr, phantom, "\mathrm{IH}"] & b \arrow[d, dashed, two heads] \\
d \arrow[r, dashed, two heads] \arrow[d] & e \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & f
\end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$: let $c \;{}^{n+1}\!\!\leftarrow a \rightarrow^* b$.

  Then $c \leftarrow d \;{}^{n}\!\!\leftarrow a \rightarrow^* b$ for some d, and:

  - By induction hypothesis $d \rightarrow^* e \;{}^{*}\!\!\leftarrow b$.
  - Then $c \rightarrow^* f \;{}^{*}\!\!\leftarrow e$ since $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

  Hence $c \rightarrow^* \cdot {}^{*}\!\!\leftarrow b$.

∎

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN $\qquad \circlearrowleft a \longrightarrow b$
3. CR $\Longrightarrow$ NF $\Longrightarrow$ UN $\Longrightarrow$ $\mathrm{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$ $\qquad a \longleftarrow b \longrightarrow \overset{\circlearrowleft}{c} \longleftarrow d \longrightarrow e$
5. NF $\not\Longleftarrow$ UN $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$
$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$
$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$
$$\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN $\qquad \circlearrowright a \longrightarrow b$
3. CR $\Longrightarrow$ NF $\Longrightarrow$ UN $\Longrightarrow$ $\mathrm{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$ $\qquad a \longleftarrow b \longrightarrow \overset{\circlearrowleft}{c} \longleftarrow d \longrightarrow e$
5. NF $\not\Longleftarrow$ UN $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. CR $\not\Longleftarrow$ NF $\qquad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \;\cancel{\Longleftarrow}\; \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \;\cancel{\Longleftarrow}\; \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \;\cancel{\Longleftarrow}\; \mathsf{UN}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \;\cancel{\Longleftarrow}\; \mathsf{NF}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \;\cancel{\impliedby}\; \mathsf{WN}$ $\qquad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \;\cancel{\impliedby}\; \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \;\cancel{\impliedby}\; \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \;\cancel{\impliedby}\; \mathsf{NF}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow} \iff {\leftrightarrow^*} = {\downarrow} \iff {\leftarrow \cdot \rightarrow^*} \subseteq {\downarrow}$
8. $\mathsf{WN}\ \&\ \mathsf{UN}^{\rightarrow} \implies \mathsf{CR}$

Definitions

- WCR local confluence or weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$

Definitions

- WCR local confluence or weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \ \subseteq \ \downarrow$
  - $\forall a, b, c$ $\exists d$

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted, two heads] & & c \arrow[dl, dotted, two heads] \\
 & d &
\end{tikzcd}
$$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN} \;\&\; \mathsf{UN}^{\rightarrow} \Longrightarrow \mathsf{CR}$
9. $\mathsf{CR} \Longrightarrow \mathsf{WCR}$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. $\mathsf{CR} \Longleftrightarrow \leftrightarrow^* \subseteq \downarrow \Longleftrightarrow \leftrightarrow^* = \downarrow \Longleftrightarrow \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN}\ \&\ \mathsf{UN}^{\rightarrow} \Longrightarrow \mathsf{CR}$
9. $\mathsf{CR} \Longrightarrow \mathsf{WCR}$
10. $\mathsf{CR} \not\Longleftarrow \mathsf{WCR}$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\quad$ $\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \arrow[loop right] \end{tikzcd}$
7. $\mathsf{CR} \Longleftrightarrow \;\leftrightarrow^* \subseteq \downarrow \;\Longleftrightarrow\; \leftrightarrow^* = \downarrow \;\Longleftrightarrow\; \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN}\ \&\ \mathsf{UN}^{\rightarrow} \Longrightarrow \mathsf{CR}$
9. $\mathsf{CR} \Longrightarrow \mathsf{WCR}$
10. $\mathsf{CR} \not\Longleftarrow \mathsf{WCR}$ $\quad$ $\begin{tikzcd} a & b \arrow[l] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d \end{tikzcd}$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \;\not\Longleftarrow\; \mathsf{WN}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \;\not\Longleftarrow\; \mathsf{UN}^{\rightarrow}$ $\quad$ $\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$
5. $\mathsf{NF} \;\not\Longleftarrow\; \mathsf{UN}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$
6. $\mathsf{CR} \;\not\Longleftarrow\; \mathsf{NF}$ $\quad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \arrow[loop right] \end{tikzcd}$
7. $\mathsf{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow} \iff {\leftrightarrow^*} = {\downarrow} \iff {\leftarrow \cdot \rightarrow^*} \subseteq {\downarrow}$
8. $\mathsf{WN} \;\&\; \mathsf{UN}^{\rightarrow} \implies \mathsf{CR}$
9. $\mathsf{CR} \implies \mathsf{WCR}$
10. $\mathsf{CR} \;\not\Longleftarrow\; \mathsf{WCR}$ $\quad$ $\begin{tikzcd} a & b \arrow[l] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d \end{tikzcd}$
11. $\mathsf{SN} \;\&\; \mathsf{WCR} \implies \mathsf{CR}$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. $\mathsf{CR} \Longleftrightarrow \leftrightarrow^* \subseteq \downarrow \Longleftrightarrow \leftrightarrow^* = \downarrow \Longleftrightarrow \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN}\ \&\ \mathsf{UN}^{\rightarrow} \Longrightarrow \mathsf{CR}$
9. $\mathsf{CR} \Longrightarrow \mathsf{WCR}$
10. $\mathsf{CR} \not\Longleftarrow \mathsf{WCR}$ $\qquad a \longleftarrow b \rightleftarrows c \longrightarrow d$
11. $\mathsf{SN}\ \&\ \mathsf{WCR} \Longrightarrow \mathsf{CR}$ $\qquad$ Newman's Lemma

Summary

$$
\begin{tikzcd}
\mathsf{SN} \arrow[rrrr] \arrow[drr, bend right] & & & & \mathsf{WN} \arrow[dll, bend left] & \\
 & \& & \mathsf{CR} \arrow[dll, bend left] \arrow[dr, bend right] & & \& & \\
\mathsf{WCR} \arrow[urr, bend left] & & & \mathsf{NF} \arrow[r] & \mathsf{UN} \arrow[r] & \mathsf{UN}^{\to} \arrow[ulll, bend right]
\end{tikzcd}
$$

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$$\begin{tikzcd}
\mathrm{SN} \arrow[rrrr] \arrow[drr, bend right] & & & & \color{red}{\mathrm{WN}} \arrow[dll, bend left] & \\
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\end{tikzcd}$$

Definitions

- semi-completeness
  - CR & WN

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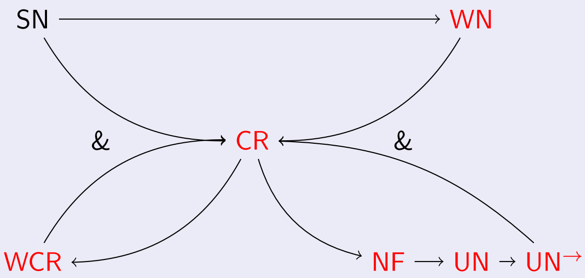

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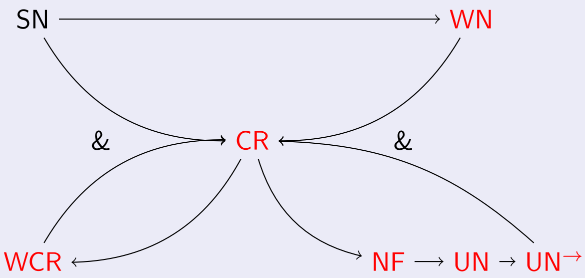

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- diamond property
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$

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b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\
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Proof.

Exercise. ∎

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ARS $\mathcal{A} = \langle A, \to_1 \rangle$ *is confluent if*

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$\implies \quad {}_1^*\!\leftarrow \cdot \to_1^* \ \subseteq \ \to_1^* \cdot {}_1^*\!\leftarrow$ ∎

Outline

Well-Founded Induction

given

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$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathsf{P}(b)\big) \implies \mathsf{P}(a)\Big) \quad \implies \quad \forall a\colon \mathsf{P}(a)$$

Newman's Lemma

$$\mathsf{SN}(\mathcal{A}) \ \&\ \mathsf{WCR}(\mathcal{A}) \quad \Longrightarrow \quad \mathsf{CR}(\mathcal{A})$$

Proof.

$$\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
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b & & & & c
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$$
\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
 & b_1 \arrow[dl, two heads] \arrow[dr, two heads, red] & \color{red}{\mathsf{WCR}} & c_1 \arrow[dl, two heads, red] \arrow[dr, two heads] & \\
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b \arrow[dr, ->>, red] & \color{red}{\mathrm{IH}} & d_1 \arrow[dl, ->>, red] & & c \\
 & \color{red}{d_2} & & &
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induction hypothesis $\mathrm{CR}(b_1)$

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b \arrow[dr, ->>] \arrow[rr, phantom, "\mathrm{IH}"] & & d_1 \arrow[dl, ->>] & & \color{red}{c} \\
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 b \arrow[dr, ->>] & \mathrm{IH} & d_1 \arrow[dl, ->>] & & c \arrow[ddll, ->>, red] \\
 & d_2 \arrow[dr, ->>, red] & & \color{red}\mathrm{IH} & \\
 & & \color{red} d_3 & &
\end{tikzcd}
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■

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  if $\forall b, c.\ (c \leftarrow a \to b \ \Rightarrow \ \exists d.\ c \to^* d \ {}^*\!\!\leftarrow b)$

- UN unique normal forms

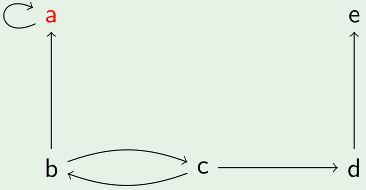
  if $\forall b, c.\ (b, c \text{ normal forms } \wedge c \leftrightarrow^* a \leftrightarrow^* b \ \Rightarrow \ b = c)$

An ARS has the property if all its elements have the respective property.

Example

$$
\begin{tikzcd}
a \arrow[loop left] & & e \\
b \arrow[u] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d \arrow[u]
\end{tikzcd}
$$

Example



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CR,WCR

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\text{WN,WCR} & &
\end{tikzcd}
$$

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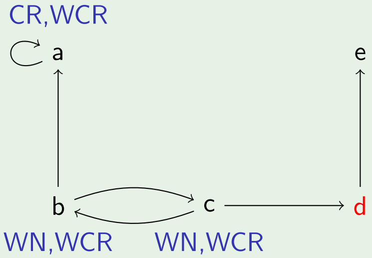

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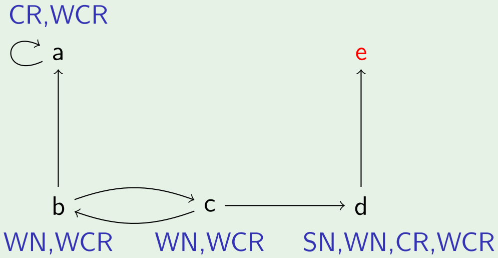

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