3 Monotonic algebras

1. From [2][Example 5.16]

   Use a linear interpretation to show termination of the rewrite rule

   $$\mathsf{a}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))$$

2. From [2][Example 5.17].

   Use a linear interpretation to show termination of the rewrite rule

   $$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))$$

3. Use a linear interpretation to show termination of the following TRS

   $$\begin{array}{rcl} \mathsf{length}(\mathsf{nil}) & \to & \mathsf{0} \\ \mathsf{length}(\mathsf{cons}(x, y)) & \to & \mathsf{s}(\mathsf{length}(y)) \end{array}$$

4. Can we use a linear interpretation to show termination of the following TRS?

   $$\begin{array}{rcl} \mathsf{add}(\mathsf{0}, y) & \to & y \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(y, x)) \end{array}$$

5. Can we use a linear interpretation to show termination of the following TRS?

   $$\begin{array}{rcl} \mathsf{add}(x, \mathsf{0}) & \to & x \\ \mathsf{add}(x, \mathsf{s}(y)) & \to & \mathsf{add}(\mathsf{s}(x), y) \\ \mathsf{mul}(x, \mathsf{0}) & \to & \mathsf{0} \\ \mathsf{mul}(x, \mathsf{s}(y)) & \to & \mathsf{add}(x, \mathsf{mul}(x, y)) \end{array}$$

6. Partially from a paper by Hans Zantema.

   Can we use polynomial interpretations to prove termination of the following TRSs?

   (a)

   $$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{a}(\mathsf{g}(\mathsf{f}(x)), x)$$

   (b)

   $$\begin{array}{rcl} \mathsf{f}(\mathsf{f}(x)) & \to & \mathsf{g}(x) \\ \mathsf{g}(\mathsf{g}(x)) & \to & \mathsf{f}(x) \end{array}$$

   (c)

   $$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{0})))$$

   (d)

   $$\begin{array}{rcl} \mathsf{f}(\mathsf{g}(x)) & \to & \mathsf{f}(\mathsf{h}(\mathsf{f}(x))) \\ \mathsf{g}(\mathsf{g}(x)) & \to & \mathsf{f}(\mathsf{g}(\mathsf{h}(x))) \end{array}$$

7. Consider the TRS with a single rule:

   $$\mathsf{f}(x) \to \mathsf{g}(\mathsf{f}(x))$$

   We try to orient this TRS using a polynomial interpretation:

- $[\mathsf{f}](x) = x + 1$
- $[\mathsf{g}](x) = 0$

(a) Do we have $[\![\mathsf{f}(x)]\!] > [\![\mathsf{g}(\mathsf{f}(x))]\!]$?

(b) Is this TRS terminating?

(c) Why was this a trick question?

8. From [5][Example 5.3.5 and Example 5.3.9].

Consider two binary function symbols $\oplus$ and $\odot$. We consider $A = \mathbb{N} - \{0, 1\}$. We define the following interpretations of the function symbols:

$$\begin{array}{rcl} [\oplus](x, y) & = & xy \\ [\odot](x, y) & = & 2x + y + 1 \end{array}$$

(a) Show that for any $x, y \in A$ we have both $[\oplus](x, y) \in A$ and $[\odot](x, y) \in A$.

(b) Show that the above interpretation is monotone.

(c) Show that the above interpretation can be used to show termination of the TRS with the following rewrite rules:

$$\begin{array}{rcl} x \odot (y \oplus z) & \rightarrow & (x \odot y) \oplus (x \odot z) \\ (x \oplus y) \oplus z & \rightarrow & x \oplus (y \oplus z) \end{array}$$

9. Use cost-size interpretations to show termination of

$$\begin{array}{rcl} \mathsf{a}(\mathsf{b}(x)) & \rightarrow & \mathsf{a}(\mathsf{c}(x)) \\ \mathsf{a}(\mathsf{c}(x)) & \rightarrow & \mathsf{b}(\mathsf{b}(\mathsf{a}(x))) \end{array}$$

4 Path orderings

1. Use the (iterative) lexicographic path order to prove termination of

$$c(b(x)) \to b(b(c(x))) \qquad\qquad a(c(x)) \to c(a(x))$$

2. Use the (iterative) lexicographic path order to prove termination of

$$\begin{aligned} f(x \cdot y) &\to f(x) + y \\ f(x + y) &\to f(x) + f(y) \\ (x + y) \cdot z &\to (x \cdot z) + (y \cdot z) \end{aligned}$$

3. Let $>$ be the usual order on $\mathbb{N}$ extended to $\mathbb{N} \cup \{ a, b, c, d \}$ by

$$a < b < c < d \ ,$$

leaving numbers and letters incomparable to each other.

Decide for each pair of multisets whether $M \succ_{mul} N$, $M \prec_{mul} N$, or M and N are incomparable:

(a) $M = \{5, 3, 3, 1\}$, $N = \{4, 4, 4, 4\}$

(b) $M = \{4, 1, d, d\}$, $N = \{3, 3, 3, 2, c, c, c, d\}$

(c) $M = \{3, 1, b, c\}$, $N = \{3, c, c\}$

4. Use the *recursive* path order to prove termination of

$$f(s(x), y) \to f(y, x)$$

5. Use the (iterative) lexicographic path order to prove termination of

$$a(b(x, y)) \to b(a(x), b(a(y), x))$$

6 Lambda calculus

1. Draw the term tree of the following λ-term, and indicate which variables are bound, and by which λ they are bound:

   $(\lambda xy.\, x\, x\, y)\, (\lambda zu.\, u\, (u\, z))\, \lambda w.\, w.$

2. Compute the following substitutions:

   (a) $(x\, y)[y := u\, w]$
   (b) $(x\, y)[x := u\, w]$
   (c) $(\lambda x.\, x\, y)[y := x\, x]$
   (d) $((\lambda x.\, x)\, x\, (\lambda y.\, x\, u))[x := y]$

3. Reduce $(\lambda x.\, x\, x)\, (\lambda s.\lambda z.\, s\, z)$ to β-normal form.

4. Reduce the term $\Omega = (\lambda x.\, x\, x)\, (\lambda x.\, x\, x)$.

5. The term $\mathsf{Y} = \lambda f.\, (\lambda x.\, f\, (x\, x))\, (\lambda x.\, f\, (x\, x))$ is Curry's fixed point combinator.

   A definition: $x \in A$ is a fixed point of $f : A \to B$ if $f(x) = x$. For λ-calculus, for equality we use β-equality which is β-conversion (so any number of steps, from left to right or from right to left). The definition for λ-calculus: M is a fixed point of F if $F\, M =_\beta M$.

   We can use Y to construct a fixed point for any term.

   (a) Show that: $F\, (\mathsf{Y}\, F) =_\beta \mathsf{Y}\, F$ for every λ-term F.

   (b) We consider the λ-term $M = \mathsf{Y}\, (\lambda a.\, \lambda b.\, c)$.
   Give a β-reduction that starts in M and ends in a β-normal form, and give an infinite reduction that starts in M.

6. Reduce the following terms:

   (a) $(\lambda x.\, x)\, (\lambda y.\, y\, z)\, a$
   (b) $(\lambda x.\, \lambda y.\, f\, x\, y)\, a\, b$
   (c) $(\lambda x.\, x\, a)\, (\lambda y.\, y\, b)$

   This illustrates the three ways in which β-redexes can be created.

7. Give in a picture a term that has four redexes: a redex that is both outermost and innermost, an outermost-redex that is not innermost, a redex that is neither outermost nor innermost, and a redex that is innermost but not outermost.