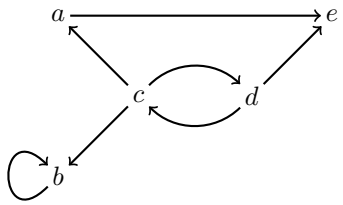
ISR 2026
Exercises Introduction to Term Rewriting

Contents

1 Abstract Reduction Systems (ARSs)

1. From [4].

   Consider the following ARS:



   Complete the following table, indicating whether a property holds yes or no for an element.

	SN	WN	$\text{UN}^{\rightarrow}$	CR	WCR
a					
b					
c					
d					
e					

2. From [4].

   We consider an object a in an ARS $\mathcal{A} = (A, \rightarrow)$.

   (a) Which of the following implications hold?
   - i. $\text{SN}(a) \Rightarrow \text{WN}(a)$
   - ii. $\text{CR}(a) \Rightarrow \text{WCR}(a)$
   - iii. $\text{CR}(a) \Rightarrow \text{UN}^{\rightarrow}(a)$

   (b) Show that the implication $\text{WN}(a)$ & $\text{UN}^{\rightarrow}(a) \Rightarrow \text{CR}(a)$ does not hold in general.

   (c) Does the implication $\text{WN}(a)$ & $\text{UN}^{\rightarrow}(\mathcal{A}) \Rightarrow \text{CR}(a)$ hold?

3. Give an ARS that is $\mathsf{CR}^{\leq 1}$ and WN but not SN.

   $\mathsf{CR}^{\leq 1}$ is the property that $\leftarrow \cdot \rightarrow \subseteq \rightarrow^{=} \cdot \leftarrow^{=}$.

4. From [3].

   Give an ARS that is WCR and $\mathsf{UN}^{\rightarrow}$ but not UN.

5. From [3].

   Show that CR implies NF.

6. From [3].

   Give an ARS that is NF but not CR.

7. From [3].

   Give an ARS that is UN but not CR.

8. From [3].

   Show that WN and $\mathsf{UN}^{\rightarrow}$ together imply CR.

9. The following is often used in proofs of confluence. Let $(A, \rightarrow)$ be an ARS and let $\rightsquigarrow$ be a confluent reduction relation on A. If we have $\rightarrow \subseteq \rightsquigarrow \subseteq \rightarrow^*$ then $\rightarrow$ is confluent.

10. (Difficult) From [3].

    Show that CR^1 and WN together imply SN. CR^1 is the property that $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$.

2 Term Rewriting Systems (TRSs)

1. From [6].

   Give the reduction graph of

   $$\mathsf{mul}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))), \mathsf{s}(\mathsf{s}(0)))$$

2. From [2][Section 5.1], also [6][p17].

   Give an infinite reduction sequence in the String Rewriting System (SRS) $\mathsf{ba} \to \mathsf{aabb}$.

3. Show that the SRS $\mathsf{aa} \to \mathsf{aba}$ is terminating.

4. From [2][Exercise 5.5].

   Give a TRS consisting of one rewrite rule that is WN but not SN.

5. find a natural signature and type assignment for the following many-sorted TRS. Use at least two sorts.

$$\begin{array}{rcl}
\mathsf{add}(\mathsf{pos}(x), \mathsf{pos}(y)) & \to & \mathsf{pos}(\mathsf{plus}(x, y)) \\
\mathsf{add}(\mathsf{neg}(x), \mathsf{neg}(y)) & \to & \mathsf{neg}(\mathsf{s}(\mathsf{plus}(x, y))) \\
\mathsf{plus}(0, y) & \to & y \\
\mathsf{subtract}(x, 0) & \to & \mathsf{pos}(x) \\
\mathsf{subtract}(0, \mathsf{s}(y)) & \to & \mathsf{neg}(y) \\
\mathsf{subtract}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{subtract}(x, y)
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{add}(\mathsf{pos}(x), \mathsf{neg}(y)) & \to & \mathsf{subtract}(x, \mathsf{s}(y)) \\
\mathsf{add}(\mathsf{neg}(x), \mathsf{pos}(y)) & \to & \mathsf{subtract}(y, \mathsf{s}(x)) \\
\mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{plus}(x, \mathsf{s}(y))
\end{array}$$

6. Consider the following term rewriting system

$$\begin{array}{lrcl}
(1) & \mathsf{f}(x, x) & \to & \mathsf{b} \\
(2) & \mathsf{f}(x, \mathsf{g}(y)) & \to & \mathsf{f}(\mathsf{g}(x), y) \\
(3) & \mathsf{g}(\mathsf{f}(x, y)) & \to & \mathsf{f}(\mathsf{g}(y), \mathsf{g}(x)) \\
(4) & \mathsf{g}(\mathsf{g}(x)) & \to & \mathsf{g}(x)
\end{array}$$

   Rewrite the term $\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{a})), \mathsf{a}))$ to normal form, first using the innermost strategy, then using the outermost strategy.